Exercise set 8

As in the lectures we denote by Ab the category of abelian groups, by Ab_X the category of sheaves (of abelian groups) on a topological space X, and by PAb_X the category of presheaves of abelian groups on X. For each topolgical space X and each abelian group A we denote by $H^i(X, A)$ the i-th sheaf cohomology group of X with coefficients in the constant sheaf $\underline{A}_X$.

First, some technicalities:

Exercise 1. Let V be a $\mathbf{C}$-vector space and let $I\colon V \to V$ be a linear endomorphism such that $I^2 = -1$. Show that $V = V^{(1,0)} \oplus V^{(0,1)}$ where $V^{(1,0)}$ is the i-eigenspace of I and $V^{(0,1)}$ is the $(-i)$-eigenspace. NB: We *do not* assume that $\dim V < \infty$.

Next we move to differential forms:

Exercise 2. Let X be a Riemann surface.

(a) Check that the endomorphism $I\colon \mathcal{A}^1_X \to \mathcal{A}^1_X$ sends the subsheaf $\mathcal{F}^1_X$ of real differential 1-forms to itself.

Next, as X is a smooth manifold, for each point $x \in X$ we have the tangent space T_xX that is an $\mathbf{R}$-vector space of dimension 2. Every real 1-form $\omega \in \mathcal{F}^1_X(X)$ gives rise to a linear functional $\omega|_x\colon T_xX \to \mathbf{R}$. Further, in Exercise 4 of the first set we equipped T_xX with a compatible structure of a $\mathbf{C}$-vector space of dimension 1.

(b) Prove that for each real 1-form $\omega \in \mathcal{F}^1_X(X)$ and each point $x \in X$ we have an equality

$$I(\omega)\Big|_x = \omega\Big|_x \circ J_x$$

where $J_x\colon T_xX \to T_xX$ is the multiplication by i.

Exercise 3. Let S^2 be the Riemann sphere. Show that $H^0(S^2, \Omega_{S^2}) = 0$.

Exercise 4. Let $f\colon X \to Y$ be a morphism of Riemann surfaces. By calculus on manifolds we have pullback morphisms $f^*_{\mathrm{dR}}\colon \mathcal{A}^p_Y(Y) \to \mathcal{A}^p_X(X)$ for all $p \geqslant 0$.

(a) Show that for $p = 1$ the pullback f^*_{dR} transforms holomorphic differential forms to holomorphic differential forms. Varying an open $U \subset Y$ construct a morphism of sheaves $f^*_{\mathrm{dR}}\colon \Omega_Y \to f_*\Omega_X$. (In the case $p = 0$ the same process gives the usual morphism $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.)

(b) Assuming that X is connected and f is not constant show that the pullback morphism $f^*_{\mathrm{dR}}\colon H^0(Y, \Omega_Y) \hookrightarrow H^0(X, \Omega_X)$ is an injection.

Exercise 5. Let $\Lambda \subset \mathbf{C}$ be a lattice and consider the quotient Riemann surface $E := \mathbf{C}/\Lambda$.

(a) Prove that $\dim H^0(E,\, \Omega_E) = 1$. Hint: The projection $\pi\colon \mathbf{C} \twoheadrightarrow E$ is a morphism of Riemann surfaces and so induces a pullback map $\pi^*_{\mathrm{dR}}\colon H^0(E,\, \Omega_E) \to H^0(\mathbf{C},\, \Omega_{\mathbf{C}})$. Can you describe the image?

(b) Deduce that every morphism of Riemann surfaces $S^2 \to E$ is constant.

Here are some more exercises concerning homological algebra:

Exercise 6. Let $a > 0$ be an integer and consider the left exact functor

$$F(-) := \mathrm{Hom}(\mathbf{Z}/a\mathbf{Z},\ -)\colon\ Ab \longrightarrow Ab\,.$$

(a) Compute $R^nF(\mathbf{Z})$ for all $n \geqslant 0$.

(b) Compute $R^nF(\mathbf{Z}/a\mathbf{Z})$ for all $n \geqslant 0$.

Exercise 7. Let $\mathcal{A}$ be an abelian category and let $F\colon Ab \to \mathcal{A}$ be a left exact functor. Prove that for each abelian group A and each integer $n \geqslant 2$ we have $R^nF(A) = 0$.

Exercise 8. Let k be a field, and let $V^\bullet$ be a complex of k-vector spaces. Suppose that every vector space V^n is finite-dimensional, and that $V^n = 0$ for all but finitely many degrees n. Show that

$$\sum_n (-1)^n \dim V^n = \sum_n (-1)^n \dim H^n(V^\bullet).$$

Conclude that $\sum_n (-1)^n \dim V^n = 0$ whenever the complex $V^\bullet$ is exact.

Let us prove the existence of injective resolutions of sheaves. We begin with abelian groups, i.e. sheaves on the singleton space.

Exercise 9. For each abelian group A set $A^* := \mathrm{Hom}_{\mathbf{Z}}(A,\, \mathbf{Q}/\mathbf{Z})$.

(a) Prove that A^* is an injective object whenever A is torsion-free.

(b) Prove that the natural morphism $A \hookrightarrow (A^*)^*$ is injective.

(c) Deduce that the category of abelian groups has enough injectives. Hint: try to replace A^* with a torsion-free abelian group.

(d) Extra: Construct a functor $F\colon\ Ab \to Ab$ and an injective natural transformation $\mathrm{Id} \hookrightarrow F$ such that $F(A)$ is an injective object for every abelian group A. Deduce that there exists a functor $Ab \to \mathrm{Ch}(Ab)$ sending an abelian group to its injective resolution.

Exercise 10. Let X be a topological space. Prove that the category of sheaves Ab_X has enough injectives. You can use the following approach:

(a) Let $X^\sharp$ be the same set X equipped with the discrete topology. Show that the category $Ab_{X^\sharp}$ has enough injectives. Hint: Relate the category $Ab_{X^\sharp}$ to Ab and invoke Exercise 9(c).

(*b*) Let $\iota\colon X^\sharp \to X$ be the map that is the identity on the underlying sets. Show that for every sheaf $\mathcal{F}$ on X the adjunction unit $\mathcal{F} \hookrightarrow \iota_*\iota^{-1}\mathcal{F}$ is an injection. Deduce that the category Ab_X has enough injectives.

(*c*) Extra: Construct a functor $F\colon\ Ab_X \to Ab_X$ and an injective natural transformation $\mathrm{Id} \hookrightarrow F$ such that $F(\mathcal{F})$ is an injective sheaf for every sheaf $\mathcal{F}$. Deduce that there exists a functor $Ab_X \to \mathrm{Ch}(Ab_X)$ sending a sheaf to its injective resolution.

Below we will consider an arbitrary abelian category $\mathcal{A}$, but for the sake of simplicity you are free to assume that $\mathcal{A}$ is the category of (left) modules over a ring. First let us discuss the notion of a *split exact sequence.*

Exercise 11. (*a*) Consider a commutative diagram in $\mathcal{A}$ with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

Prove that the arrow h is an isomorphism.

We say that a short exact sequence

$$(*) \qquad 0 \longrightarrow A \xrightarrow{\ f\ } E \xrightarrow{\ g\ } B \longrightarrow 0$$

is *split* if there is a morphism $h\colon E \to A \oplus B$ that makes the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\ f\ } & E & \xrightarrow{\ g\ } & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{\ i\ } & A \oplus B & \xrightarrow{\ p\ } & B & \longrightarrow & 0. \end{array}$$

Here i is the natural inclusion and p is the natural projection. By part (*a*) the morphism h is necessarily an isomorphism.

(*b*) Show that $(*)$ is split if and only if there is a morphism $t\colon E \to A$ such that $tf = 1_A$.

(*c*) Show that $(*)$ is split if and only if there is a morphism $s\colon B \to E$ such that $gs = 1_B$.

(*d*) Show that additive functors transform split short exact sequences to split short exact sequences.

(*e*) Suppose that $\mathcal{A}$ is the category of vector spaces over a field. Prove that in this case every short exact sequence is split.

Exercise 12. Consider a short exact sequence of complexes in the abelian category $\mathcal{A}$:

$$(\dagger) \qquad 0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0.$$

We assume that this sequence is *termwise split*, which is to say, the sequence of objects $0 \to A^n \to B^n \to C^n \to 0$ is split for every index $n \in \mathbf{Z}$. (Importantly, this is much weaker than demanding the sequence $(\dagger)$ to be split in the category of complexes.)

(a) Pick splitting morphisms $s^n \colon C^n \to B^n$ and $t^n \colon B^n \to A^n$. Show that the family of morphisms

$$t^{n+1} \circ d_B^n \circ s^n \colon C^n \to A^{n+1}$$

defines a morphism of complexes $\delta \colon C^\bullet \to A^\bullet[1]$. Hint: try composing with suitable monomorphisms.

(b) Show that the homotopy class of δ is independent of the choice of splttings s^n, t^n.

(c) Show that the induced morphism $H^n(\delta) \colon H^n(C^\bullet) \to H^n(A^\bullet[1]) = H^{n+1}(A^\bullet)$ is exactly the boundary homomorphism in the long exact sequence of cohomology objects induced by (†)

This gives an alternative construction of the boundary homomorphism in the long exact sequence of derived functors.