Exercise set 5

As in the lectures we denote by Ab the category of abelian groups, and by Ab_X the category of sheaves (of abelian groups) on a topological space X.

First, let us recap some material of the lectures. Recall that an abelian category $\mathcal{A}$ has *enough injectives* if for every object A of $\mathcal{A}$ there exists a monomorphism $A \hookrightarrow I$ into an injective object.

Exercise 1. Suppose that an abelian category $\mathcal{A}$ has enough injectives. Show that every object of $\mathcal{A}$ admits a (right) injective resolution.

Recall that an additive functor is *left exact* if it transforms kernels to kernels, and is *right exact* if it transforms cokernels to cokernels.

Exercise 2. Let $\mathcal{A}$ be an abelian category and let A be an object of $\mathcal{A}$. Show that the functor $\mathrm{Hom}_{\mathcal{A}}(A, \ )$ from $\mathcal{A}$ to Ab is left exact.

Exercise 3. Let R be an associative unital ring. Show that a left R-module is projective if and only if it is a direct summand of a free left R-module. Hint: Every left R-module is a quotient of a free left R-module.

Exercise 4. Let X be a topological space, and let $\underline{\mathbf{Z}}_X$ be the sheaf of locally constant $\mathbf{Z}$-valued functions on X. Prove that for each sheaf $\mathcal{F}$ on X we have a natural isomorphism

$$\mathrm{Hom}(\underline{\mathbf{Z}}_X, \mathcal{F}) \ \cong \ \mathcal{F}(X).$$

Deduce that the global sections functor $\mathcal{F} \mapsto \mathcal{F}(X)$ from Ab_X to Ab is left exact. (This is an alternative to the argument given in the lectures.)

Next, here are some exercises of slightly increased difficulty:

Exercise 5. Let $n > 1$ be a positive integer, and consider the functor $F\colon\ Ab \to Ab$ that sends an abelian group A to the subgroup nA.

(a) Show that the functor F transforms monomorphisms to monomorphisms, and epimorphisms to epimorphisms.

(b) Show that the functor F is neither left nor right exact.

Exercise 6. Let $\mathcal{A}$ be an abelian category, and let $I^\bullet$ be a complex of injective objects of $\mathcal{A}$. Suppose that the complex $I^\bullet$ is

- *bounded below*, i.e. $I^n = 0$ for all $n \ll 0$, and is
- *exact*, i.e. $H^n(I^\bullet) = 0$ for all n.

Show that in the homotopy category $K(\mathcal{A})$ we have

$$I^\bullet = 0,$$

i.e. the complex $I^\bullet$ is a zero object. Hint: Construct a homotopy between the identity and the zero endomorphisms of $I^\bullet$.

Here are more exercises concerning sheaves:

Exercise 7. Let X be a Riemann surface. Let $\mathcal{F}$ be the presheaf of functions on X admitting a holomorphic logarithm, i.e. functions of the form $e \circ f$ where $f\colon U \to \mathbf{C}$ is holomorphic and $e\colon \mathbf{C} \to \mathbf{C}^\times$ is the exponential function. Important: The addition of sections of $\mathcal{F}$ is given by *pointwise multiplication* of functions.

(*a*) Show that there is a short exact sequence of *presheaves* of the form

$$0 \longrightarrow \underline{\mathbf{Z}}_X \xrightarrow{f\mapsto 2\pi i f} \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow 0.$$

(*b*) Show that $\mathcal{F}$ is not a sheaf.

(*c*) (extra) Let $\mathcal{O}_X^\times$ be a presheaf on X defined by the formula

$$\mathcal{O}_X^\times(U) := \{\ f\colon U \to \mathbf{C}^\times \mid f \text{ holomorphic }\}.$$

As in the case of $\mathcal{F}$ the addition of sections is given by pointwise multiplication of functions. Show that $\mathcal{O}_X^\times$ is the sheafification of $\mathcal{F}$.

Exercise 8. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a topological space X. Consider a presheaf on X given by the formula

$$U \mapsto \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$$

where the Hom is taken in the category of sheaves on U. Show that this presheaf is a sheaf.

Remark. The sheaf in Exercise 8 is called the *Hom sheaf* and is denoted by $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})$.

Finally, here are some more exercises on abelian categories:

Exercise 9. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be an equivalence of categories. We *do not* assume a priori that F is additive.

(*a*) Show that the functor F is additive. Hint: Use Exercise 1 from the previous set.

(*b*) Show that the functor F is exact, i.e. transforms short exact sequences to short exact sequences. Hint: Deduce separately that F preserves kernels and cokernels.

Exercise 10. Let k be a field and let Vec_k be the category of k-vector spaces. Recall that a *graded k-vector space* is a k-vector space V equipped with a direct sum decomposition $V = \bigoplus_{i\in\mathbf{Z}} V^i$. A morphism of graded vector spaces is a morphism of underlying k-vector spaces that preserves the grading. Let GrVec_k be the category of graded k-vector spaces. Show that the functor

$$F\colon K(\mathrm{Vec}_k) \longrightarrow \mathrm{GrVec}_k, \quad A^\bullet \longmapsto \bigoplus_{i\in\mathbf{Z}} H^i(A^\bullet)$$

is an equivalence of categories. Deduce that the homotopy category $K(\mathrm{Vec}_k)$ is abelian. (This is essentially the only case in which the homotopy category happens to be abelian.)