Exercise set 2

Here is an exercise I forgot to add to the first set:

Exercise 1. Consider the complex manifold $\mathbf{C}^2 := \mathbf{C} \times \mathbf{C}$ and let U be the open subset $\mathbf{C}^2 \smallsetminus \{0\}$. Let $f\colon U \to \mathbf{C}$ be a holomorphic function such that for every $\alpha \in \mathbf{C}^\times$ and every $z \in U$ we have $f(\alpha z) = f(z)$. Prove that f is constant.

Next, let us move to sheaves.

Exercise 2. (*a*) Let X be a topological space, and let $\mathcal{F}$ be a sheaf of abelian groups on X. Show that $\mathcal{F}(\varnothing) = 0$.

(*b*) Suppose that the topological space X consists of a single point. Prove that the functor $\mathcal{F} \mapsto \mathcal{F}(X)$ induces an equivalence of categories of sheaves of abelian groups on X and of abelian groups.

Next, recall two general constructions for sheaves. First, let $\mathcal{F}$ be a sheaf on a topological space X. For every open subset $U \subset X$ we then have the *restriction* $\mathcal{F}|_U$, defined by the formula

$$(\mathcal{F}|_U)(V) := \mathcal{F}(V).$$

Second, given a continuous map of topological spaces $f\colon X \to Y$ and a sheaf $\mathcal{F}$ on X we define the *pushforward* $f_*\mathcal{F}$ by the formula

$$(f_*\mathcal{F})(U) := \mathcal{F}\big(f^{-1}(U)\big).$$

Recall that for each topological space we denote by $\mathcal{C}_X$ the sheaf of continuous $\mathbf{C}$-valued functions on X. This is a sheaf of rings, more precisely, a sheaf of $\mathbf{C}$-algebras. Given a continuous map $f\colon X \to Y$ we have a morphism of sheaves $f^\sharp\colon \mathcal{C}_Y \to f_*\mathcal{C}_X$ sending a function $h \in \mathcal{C}_Y(U)$ to the function

$$f^\sharp(h) := h \circ f|_{f^{-1}(U)}.$$

By construction the composite $h \circ f|_{f^{-1}(U)}$ is an element of

$$(f_*\mathcal{O}_X)\big(U\big) := \mathcal{O}_X\big(f^{-1}(U)\big)$$

so the morphism of sheaves $f^\sharp$ is well-defined. For each open $U \subset Y$ the morphism $f^\sharp\colon \mathcal{C}_Y(U) \to (f_*\mathcal{C}_X)(U)$ is a homomorphism of rings.

Recall also that a *space with a sheaf of functions* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a subsheaf $\mathcal{O}_X \subset \mathcal{C}_X$ such that for every open $U \subset X$ the subset $\mathcal{O}_X(U) \subset \mathcal{C}_X(U)$ is a subring. A *morphism* of spaces with sheaves of functions

$f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f\colon X \to Y$ such that for every open $U \subset Y$ we have an inclusion

$$f^\sharp\big(\mathcal{O}_Y(U)\big) \subset (f_*\mathcal{O}_X)(U).$$

Let $(X, \mathcal{O}_X)$ be a space with a sheaf of functions. Then every open subset $U \subset X$ has an induced structure of a space with a sheaf of functions $(U, \mathcal{O}_X|_U)$.

For every Riemann surface X we have the *structure sheaf* $\mathcal{O}_X$ defined by the formula

$$\mathcal{O}_X(U) := \{\ f\colon U \to \mathbf{C} \mid f \text{ is holomorphic }\}.$$

This is naturally a subsheaf of rings of $\mathcal{C}_X$. Consequently, the pair $(X, \mathcal{O}_X)$ is a space with a sheaf of functions. But in fact much more is true, as elaborated in the next exercise:

Exercise 3. Prove that the construction $X \mapsto (X, \mathcal{O}_X)$ is a functor from the category of Riemann surfaces to the category of spaces with sheaves of functions. Prove further that the functor $X \mapsto (X, \mathcal{O}_X)$ defines an equivalence of categories of Riemann surfaces, and of spaces with a sheaf of functions $(X, \mathcal{O}_X)$ that have the following additional properties:

(a) The underlying topological space X is Hausdorff.

(b) Each point $x \in X$ has an open neighbourhood U such that the object $(U, \mathcal{O}_X|_U)$ is isomorphic to a space with a sheaf of functions of the form $(V, \mathcal{O}_V)$ where V is an open subset of $\mathbf{C}$.

Hint: Show that the functor $X \mapsto (X, \mathcal{O}_X)$ is fully faithful and essentially surjective. Note also that the structure of a Riemann surface is given by an equivalence class of atlases.

If you feel confident with Exercise 3, you can also do the following two variants:

Exercise 4. Formulate and prove an analog of Exercise 3 for complex manifolds.

Exercise 5. Formulate and prove an analog of Exercise 3 for smooth manifolds. Here you will need the sheaf $\mathcal{C}_{X,\mathbf{R}}$ of continuous $\mathbf{R}$-valued functions on a topological space X, and the corresponding notion of a *space with a sheaf of* $\mathbf{R}$*-valued functions.* Each smooth manifold M carries a sheaf $\mathcal{C}^\infty_M$ of smooth $\mathbf{R}$-valued functions that is naturally a subsheaf of rings in $\mathcal{C}_{X,\mathbf{R}}$, and the construction $M \mapsto (M, \mathcal{C}^\infty_M)$ defines an equivalence of categories of smooth manifolds and of spaces with a sheaf of $\mathbf{R}$-valued functions with suitable extra properties.