Risk and Environmental Sustainability: Dummy Examination

11 March 2011

Instructions: The time allotted for the examination is 180 minutes. You may answer in either English or French. No written material may be brought into the examination. Full marks may be obtained with complete answers to four questions. The final mark will be based on the best four solutions.

First name:

Last name:

SCIPER number:

Exercise	Marks	Indicative marks
1		10
2		10
3		10
4		10
5		10
Total:		40

Solution 1

(a) [3, seen] Slides 51 and 54.

(b) [4, unseen] We first check the conditions of the mapping theorem. With $\mathcal{A}_q = \{(r,w) : rw = q\}$,

$$\mu^*(\{q\}) = \int_{\mathcal{A}_q} \dot\mu(r,w)\,\mathrm{d}w\mathrm{d}r = 0,$$

since the integral is over a set of measure zero. Now if $\mathcal{A}^* \subset \mathcal{E}^*$ is compact, then $\mathcal{A}^* \subset [a,b]^D$ for some $0 < a < b < \infty$, and

$$\begin{aligned}
\mu^*(\mathcal{A}^*) &\le \mu^*([a,b]^D) \\
&\le \mu\left(\{(r,w) : a \le rw_d, d = 1,\ldots,D\}\right) \\
&= \mu\left(\left\{(r,w) : a/\min_d w_d \le r\right\}\right) \\
&= \int \dot\nu(w) \int_{a/\min w_d}^{\infty} r^{-2}\mathrm{d}r\,\mathrm{d}w \\
&= a^{-1}\mathrm{E}\left(\min_d W_d\right) \\
&\le a^{-1}\mathrm{E}(W_1)
\end{aligned}$$

is finite, because $\mathrm{E}(W_1) = 1$. Hence the mapping theorem applies. The new mean measure μ^* is simply part of the statement of the theorem.

(c) [3, unseen] The intensity of $\mathcal{P}^*$ is

$$\dot\mu^*(q) = \int_{\mathcal{A}_q} \dot\mu(r,w)\,\mathrm{d}r = \int_{\mathcal{A}_q} \dot\mu(r,q/r)\left|\frac{\partial w}{\partial q}\right|\,\mathrm{d}r$$

so we need the Jacobian matrix

$$\frac{\partial q}{\partial w} = \left(\frac{\partial q_d}{\partial w_{d'}}\right)_{d,d'} = rI_D;$$

note that $q_d = rw_d$. Hence $|\partial q/\partial w| = r^D$, and thus

$$\dot\mu^*(q) = \int_0^\infty r^{-2}\dot\nu(q/r)r^{-D}\,\mathrm{d}r = \int_0^\infty u^D\dot\nu(uq)\,\mathrm{d}u,$$

after setting $r = 1/u$, because $\mathrm{d}r = -u^{-2}\mathrm{d}u$.

Solution 2

(a) [3, seen] Slides 76 and 79

(b) [1, seen] Problem 1, week 5.

(c) [3, seen/unseen] The plots show clear steps due to rounding, but these are not problematic. Over the fit appears to be very good though (bonus for this comment) the top two plots ought to have confidence bands for better interpretation. Still the lower left one shows no evidence of a problem with the fit. Its shape suggests that $\xi < 0$.

(d) [3, seen/unseen] The 95% confidence interval based on the asymptotic normal distribution of the MLE is $\widehat\xi \pm 1.96v_{\xi}^{1/2} = -0.217 \pm 1.96 \times 0.064 = (0.342, -0.091)$, so there is strong evidence that $\xi < 0$. This seems reasonable, since wave heights are not unbounded, and having $\xi \ge 0$ would correspond to an (in principle) unbounded maximum height.

Solution 3

(a) [3, seen/unseen] Slides 58 and 62. The given result implies that as $n \to \infty$,

$$\left\{1 - \frac{n\mathrm{P}(X > b_n + a_n x)}{n}\right\}^n = \left\{1 - \frac{\Lambda_n(x)}{n}\right\}^n \to \exp\left\{-(1+\xi x)_+^{-1/\xi}\right\},$$

so

$$\Lambda_n(x) = n\mathrm{P}(X > b_n + a_n x) \to (1+\xi x)_+^{-1/\xi}.$$

Hence provided the denominator is positive, the conditional probability can be written as

$$\frac{\mathrm{P}\{X > b_n + a_n(u+x)\}}{\mathrm{P}(X > b_n + a_n u)} = \frac{\Lambda_n(u+x)}{\Lambda_n(u)} \to \frac{(1+\xi x + \xi u)_+^{-1/\xi}}{(1+\xi u)_+^{-1/\xi}},$$

and a little algebra gives that this is $(1+\xi x/\sigma)_+^{-1/\xi}$, where $\sigma = 1 + \xi u$ must be positive. This implies that the limiting distribution of a threshold exceedance for X, suitably rescaled, is generalized Pareto.

(b) [3, seen/unseen] This is the law of small numbers applied to the binomial variables $N_{u,n}$, which have denominator n and success probability $\mathrm{P}(X > b_n + a_n x)$, and if $\mathrm{E}(N_{u,n}) = \Lambda_n(x)$ converges to a finite limit $(1+\xi x)_+^{-1/\xi}$, the limiting variable N_u is Poisson with mean $(1+\xi x)_+^{-1/\xi}$.

(c) [2, seen] Brief description of POT analysis of threshold exceedances.

(d) [2, seen] Problem 2 of sheet 8.

Solution 4 [10, seen] Section 4.3 of the notes, plus the extra notes provided. We expect mention of the extremogram, discussion of maxima under the $D(u_m)$ condition, local dependence and clustering, the extremal index (including a little about its estimation), clustering and return levels.

Solution 5

(a) [3, seen] Slide 181. We write $z = (z_1, z_2)$ and recall that the exponent function is defined as V in the expression $G(z) = \exp\{-V(z)\}$. Having unit Fréchet margins means that

$$G(z, \infty) = \exp\{-V(z, \infty)\} = \exp(-1/z),$$

so $V(z, \infty) = 1/z$, and likewise $V(\infty, z) = 1/z$ by symmetry. Moreover G is max-stable, so there exist functions a_t and b_t such that for every $z \in (0, \infty)^2$,

$$G(b_t + a_t z) = G(z)^t, \quad t > 0.$$

But we know that if $z = (z_1, \infty)$, then $b_t = 0$ and $a_t = 1/t$, so

$$G(b_t + a_t z) = \exp\{-V(z/t)\} = G(z)^t = \exp\{-tV(z)\},$$

which gives $tV(tz) = V(z)$ for all $t > 0$ and all $z \in (0, \infty)$. Hence V is homogeneous of order -1.

(b) [2, seen/unseen] We have

$$\mathrm{P}\{\max(Z_1, Z_2) \le z\} = \mathrm{P}(Z_1 \le z, Z_2 \le z) = \exp\{-V(z, z)\} = \exp\{-V(1,1)/z\},$$

so M has a Fréchet distribution with parameter $\theta = V(1,1)$.

The two special cases have $V(1,1) = 2$ and $V(1,1) = 1$, and correspond to independent maxima and totally dependent maxima, so θ could be interpreted as 'the number of independent maxima underlying the distribution of (Z_1, Z_2)'.

(c) [1, seen] This is the probability integral transform. Using bare hands and for $0 < u < 1$, we have

$$\mathrm{P}\{F(X) \leq u\} = \mathrm{P}\{\exp(-1/X) \leq u\} = \mathrm{P}(X \leq -1/\log u) = \exp\{-1/(-1/\log u)\} = u,$$

as required.

(d) [4, unseen] The hint gives

$$\mathrm{E}(T) = 2\mathrm{E}[\max\{F(Z_1), F(Z_2)\}] - \mathrm{E}\{F(Z_1)\} - \mathrm{E}\{F(Z_2)\},$$

and we saw in (c) that $F(Z_1)$ and $F(Z_2)$ are uniform, so their expectations are $1/2$. Note also that as F is monotone, $\max\{F(Z_1), F(Z_2)\} = F\{\max(Z_1, Z_2)\} = F(M)$, so

$$\begin{aligned}
\mathrm{E}[\max\{F(Z_1), F(Z_2)\}] &= \mathrm{E}\{F(M)\} \\
&= \int_0^\infty \exp(-1/m) \times \frac{\theta}{m^2} \exp(-\theta/m)\,\mathrm{d}m \\
&= \frac{\theta}{\theta+1} \int_0^\infty \frac{\theta+1}{m^2} \exp\{-(\theta+1)/m\}\,\mathrm{d}m \\
&= \frac{\theta}{\theta+1}.
\end{aligned}$$

Hence $\mathrm{E}(T) = 2\theta/(1+\theta) - 1 = (\theta-1)/(\theta+1)$. This could be used by computing the average $\overline{t}$ of a sample $t_1, \ldots, t_n$ of values of T, and then solving the equation $\overline{t} = (\theta-1)/(\theta+1)$ to get a moment estimate of θ.