Solution 1 We write $G(x) = \exp\{-\Lambda(x)\}$, where $\Lambda(x) = \{1 + \xi(x-\eta)/\tau\}_+^{-1/\xi}$.

(a) This seems reasonable because we seek the value x_p that a block maximum will exceed with probability $1/T$.

The exact result solves $-\Lambda(x_p) = \log(1-1/T)$ and (for $\xi \neq 0$) is $x_p = \eta + \tau[\{-\log(1-/T)\}^{-\xi} - 1]/\xi$. For the approximation, note that for large T, $-\log(1-1/T) \doteq 1/T$. Substituting this into the formula for x_p gives the desired result.

(b) Now we set $F(x)^m \doteq G(x)$ for large x, so we need to solve $1 - 1/(mT) \doteq G^{1/m}(x_p)$, which gives $G(x_p) \doteq \{1 - 1/(mT)\}^m$ or the further approximation $G(x_p) \doteq e^{-1/T}$. Solving these equations gives the stated formulae. Alternatively we note that for large mT,

$$x_p = \eta + \tau\left([-m\log\{1-1/(mT)\}]^{-\xi} - 1\right)/\xi \doteq \eta + \tau[\{-m \times -1/(mT)\}^{-\xi} - 1]/\xi = \eta + \tau(T^{\xi} - 1)/\xi.$$

(c) The blocks of one week have $m = 7 \times 24$ background observations, and $T = 20 \times 52$, the number of one-week blocks in 20 years. The value of p in terms of background observations is $1/(Tm) = 1/(20 \times 52 \times 7 \times 24)$ (ignoring leap years).

The exact and approximate values from (a) are 10.03007 and 10.03103, and from (b) they are 10.03103 and 10.03103, which are all essentially equal, so the formula used is irrelevant.

Solution 2

(a) If $H(x) = 1 - (1 + \xi x/\sigma)_+^{-1/\xi}$ denotes the generalized Pareto distribution function, then

$$\begin{aligned}
\mathrm{P}(M \le x) &= \mathrm{P}\{\max(X_1,\ldots,X_N) \le x\} \\
&= \sum_{n=0}^{\infty} \mathrm{P}\{\max(X_1,\ldots,X_N) \le x \mid N = n\}\mathrm{P}(N=n) \\
&= \sum_{n=0}^{\infty} H(x)^n \lambda^n e^{-\lambda}/n! \\
&= \exp\{\lambda H(x) - \lambda\} \\
&= \exp\left\{-\lambda(1+\xi x/\sigma)_+^{-1/\xi}\right\} \\
&= \exp\left[-\{1+\xi(x-\eta)/\tau\}_+^{-1/\xi}\right],
\end{aligned}$$

where $\eta = \sigma(\lambda^{\xi} - 1)/\xi$ and $\tau = \sigma\lambda^{\xi}$. This is of GEV form, but note that $M \ge 0$, because all the X are non-negative. Hence this formula applies for $x > 0$, and there is a probability mass of $\mathrm{P}(N=0) = e^{-\lambda}$ at $x = -\infty$, unlike for the GEV.

To check this, note that if $x \le 0$, then $H(x) = 0$, so

$$\mathrm{P}(M \le x) = \exp\{\lambda H(x) - \lambda\} = \exp(-\lambda).$$

(b) Increases in the number of maxima would correspond to an increase in λ, whereas increases in the individual values would stem from changes in σ and/or ξ. So a comprehensive model in which both λ and σ were allowed to depend on time should shed light on the cause of the increase.

Solution 3

(a) Since the extremal types theorem holds, as $n \to \infty$,

$$\Lambda_n(u) = np_u = n\mathrm{P}(X_j > u_n) = n\{1 - F(b_n + a_n u)\} \to \Lambda(u),$$

so $n^{n_u} L_1 = (np_u)^{n_u} \to \Lambda(u)^{n_u}$, and $L_2 = (1-p_u)^{n-n_u} = \{1 - \Lambda_n(u)/n\}^{n-n_u} \to \exp\{-\Lambda(u)\}$.

(b) Since $\Lambda(u)h(x-u) = \{-\dot\Lambda(x)\}$, we see that

$$L_2 L_3 \to \Lambda(u)^{n_u} \prod_{j=1}^{n_u} h(x_j - u) = \prod_{j=1}^{n_u} \{-\dot\Lambda(x_j)\}$$

so

$$n^{n_u} L \to \exp\{-\Lambda(u)\} \prod_{j=1}^{n_u} \{-\dot\Lambda(x_j)\}.$$

Since the factor n^{n_u} does not depend on the parameters, inferences based on L and on the point process likelihood will be similar for large n.

Solution 4 To see why the simulation algorithm works, note that if $E \sim \exp(1)$ and $Z = 1/E$, then

$$\mathrm{P}(Z \le z) = \mathrm{P}(1/E \le z) = \mathrm{P}(E \ge 1/z) = 1 - \{1 - \exp(-1/z)\} = \exp(-1/z), \quad z > 0,$$

which is the standard Fréchet CDF.

The first block of code gives Figure 1:

```
n <- 10000; a <- 1; i <- c(1:n)
z <- 1/rexp(n+1) # independent Frechet variables
x <- pmax(a*z[i],z[i+1])/(a+1) # moving maximum series
par(mfrow=c(1,2)) # two adjacent panels for figures
plot(i,x,log="y",pch=20, cex=.25) # should see clustering of high values,  but need log axes
qqplot(z,x,log="xy",cex=.25)
```

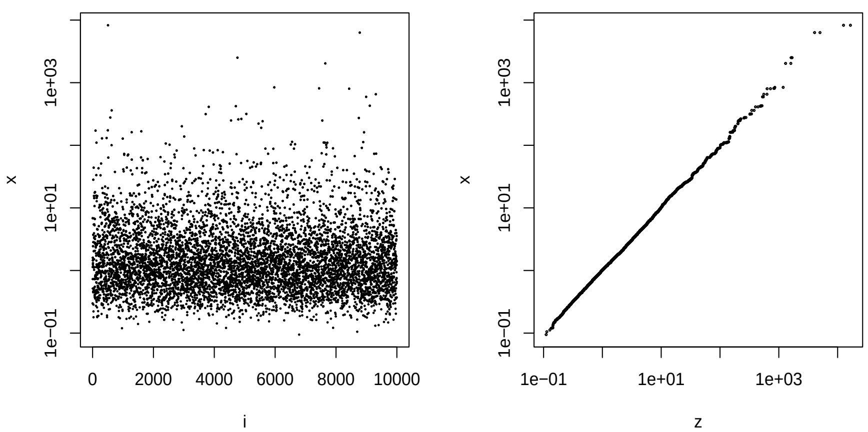

Figure 1: Plot of X (left) and QQplot of X and Z on log-scale (right).

The left-hand panel of Figure 1 may give the impression that the data are independent, but plotting the observations only in a shorter interval leads to a plot similar to that on Slide 146. The QQplot shows that the largest order statistics of X are tied, i.e., the first equals the second, the third equals the fourth and so on, because we take $a = 1$ in the formulation of X, whereas the largest order statistics of the independent process Z correspond to unique observations.

The code below results in Figure 2. Recall from Example 24 that $\theta = \max(1, a)/a + 1$, giving $\theta = 0.5$ if $a = 1$ and $\theta = 1$ if $a = 0$. We see that the estimate of θ approaches the true value as we increase the threshold.

```
tl <- quantile(z, probs = c(0.1,0.95))
exiplot(z,tl) #plots estimated theta between the limits given by tl
```

```
abline(h=1,col="red")
tl <- quantile(x, probs = c(0.1,0.95))
exiplot(x,tl)
abline(h=max(a,1)/(a+1),col="red")
```

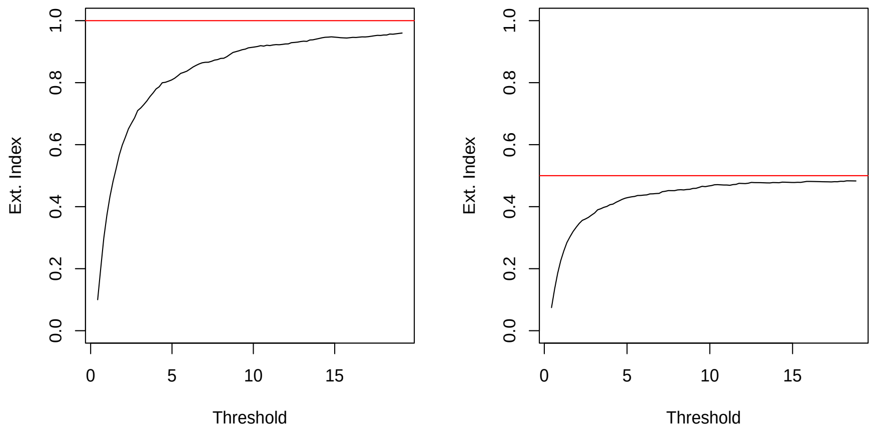

Figure 2: Estimated extremal indices for Z (left) and X (right).

We inspect the behaviour of $\widehat{\theta}_u$ for values of $a \in \{1, 1/3, 1/5, 1/7, 1/9\}$ and over different thresholds u using the code

```
par(mfrow=c(1,5))
for(a in seq(1,9, length=5)^-1){
n <- 10000; i <- c(1:n)
z <- 1/rexp(n+1) # independent Frechet variables
x <- pmax(a*z[i],z[i+1])/(a+1) # moving maximum series
tl <- quantile(x, probs = c(0.1,0.95))
exiplot(x,tl)
abline(h=max(a,1)/(a+1),col="red")
}
```

Figure 3 shows that the estimates $\widehat{\theta}_u$ behave reasonably well and that they give fairly good estimates of the extremal indices for sufficiently large values of the threshold u; these estimates are all biased downwards, however. The moving maximum model is very simplistic, so one must be cautious before reaching conclusions from the estimated $\widehat{\theta}$ in real data.

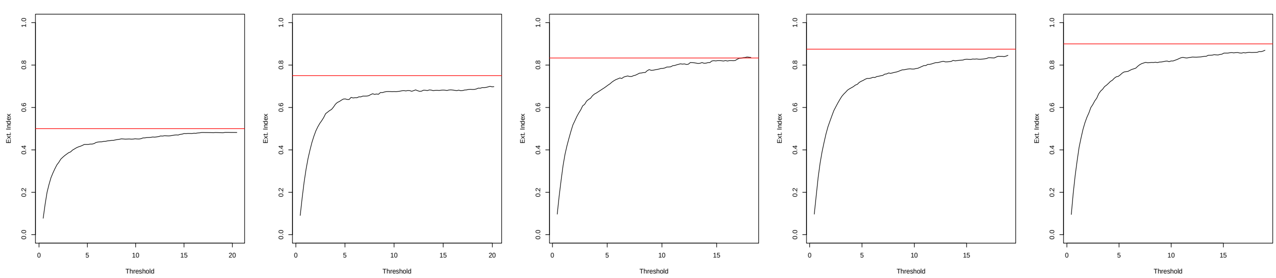

Figure 3: Left to right: Plot of $\widehat{\theta}_u$ for $a \in \{1, 1/3, 1/5, 1/7, 1/9\}$ as a function of the threshold u.