Solution 1 We follow the hint, first noting that conditional on $\mathcal{P}$ and supposing that there are events at $x_1, \ldots, x_n$,

$$\int f_1 \mathrm{d}\mathcal{P}_1 + \int f_2 \mathrm{d}\mathcal{P}_2 = \sum_{j=1}^n I(x_j) f_1(x_j) + \{1 - I(x_j)\} f_2(x_j)$$

and independence of the indicators $I(x_j)$ gives

$$\mathrm{E}\left\{\exp\left(-\int f_1 \mathrm{d}\mathcal{P}_1 - \int f_2 \mathrm{d}\mathcal{P}_2\right) \mid \mathcal{P}\right\} = \prod_{j=1}^n \left[\gamma(x_j) e^{-f_1(x_j)} + \{1 - \gamma(x_j)\} e^{-f_2(x_j)}\right].$$

Conditional on $N(\mathcal{A}) = n$, the events $X_1, \ldots, X_n$ are independently distributed on $\mathcal{A}$ with density function $\dot{\mu}(x)/\mu(\mathcal{A})$, so

$$\mathrm{E}\left\{\exp\left(-\int f_1 \mathrm{d}\mathcal{P}_1 - \int f_2 \mathrm{d}\mathcal{P}_2\right) \mid N(\mathcal{A}) = n\right\} = \left(\int_{\mathcal{A}} \left[\gamma(x) e^{-f_1(x)} + \{1 - \gamma(x)\} e^{-f_2(x)}\right] \dot{\mu}(x)\,\mathrm{d}x/\mu(\mathcal{A})\right)^n.$$

If for brevity we write B for the integral on the right, then the unconditional expectation is

$$\mathrm{E}\left\{\exp\left(-\int f_1 \mathrm{d}\mathcal{P}_1 - \int f_2 \mathrm{d}\mathcal{P}_2\right)\right\} = \sum_{n=0}^\infty \frac{\mu(\mathcal{A})^n}{n!} e^{-\mu(\mathcal{A})} \{B/\mu(\mathcal{A})\}^n = \exp\{B - \mu(\mathcal{A})\},$$

and

$$\begin{aligned}
B - \mu(\mathcal{A}) &= \int_{\mathcal{A}} \left[\gamma(x) e^{-f_1(x)} + \{1 - \gamma(x)\} e^{-f_2(x)} - 1\right] \dot{\mu}(x)\,\mathrm{d}x \\
&= \int_{\mathcal{A}} \left[\gamma(x) e^{-f_1(x)} + \{1 - \gamma(x)\} e^{-f_2(x)} - \gamma(x) - \{1 - \gamma(x)\}\right] \dot{\mu}(x)\,\mathrm{d}x \\
&= -\int_{\mathcal{A}} \left\{1 - e^{-f_1(x)}\right\} \gamma(x)\dot{\mu}(x)\,\mathrm{d}x - \int_{\mathcal{A}} \left\{1 - e^{-f_2(x)}\right\} \{1 - \gamma(x)\}\dot{\mu}(x)\,\mathrm{d}x \\
&= -\int_{\mathcal{E}} \left\{1 - e^{-f_1(x)}\right\} \gamma(x)\dot{\mu}(x)\,\mathrm{d}x - \int_{\mathcal{E}} \left\{1 - e^{-f_2(x)}\right\} \{1 - \gamma(x)\}\dot{\mu}(x)\,\mathrm{d}x,
\end{aligned}$$

where the last step applies because both $1 - e^{-f_1(x)}$ and $1 - e^{-f_2(x)}$ equal zero outside $\mathcal{A}$. We recognise the exponential of this product as the product of the Laplace functionals $\mathcal{L}_{\mathcal{P}_1}(f_1)\mathcal{L}_{\mathcal{P}_2}(f_2)$ for two independent Poisson processes on $\mathcal{E}$ with respective intensities $\gamma(x)\dot{\mu}(x)$ and $\{1 - \gamma(x)\}\dot{\mu}(x)$, as required.

For the last part, thinning a Poisson process amounts to just keeping the red points, which gives the result.

Solution 2

(a) We apply the mapping theorem. The first condition holds: for each $x \in \mathbb{R}^s$, $g^{-1}\{x\}$ is a hyperplane of dimension $s < d$, which has Lebesgue measure $\mathcal{L}(g^{-1}\{x\}) = 0$. By the existence of the intensity function $\dot{\mu}$ (so the mean measure μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$), we have

$$\mu(g^{-1}\{x\}) = \int_{g^{-1}\{x\}} \dot{\mu}(u)\mathcal{L}(\mathrm{d}u) = 0,$$

so $g(\mathcal{P})$ contains no multiple points (with probability one).

The second condition also holds. Since $\mathcal{E}$ is compact and using the definition of a Poisson process, we have, for all compact $\mathcal{A} \subset \mathcal{E}$, that

$$\mu(g^{-1}\{\mathcal{A}\}) \le \mu(\mathcal{E}) < \infty.$$

Hence $g(\mathcal{P})$ is a Poisson process on $g(\mathcal{E}) \subset \mathbb{R}^s$ with mean measure

$$\mu(\mathcal{A}) = \mu\{g^{-1}(\mathcal{A})\} = \mu(\{(x_1, \ldots, x_d) \in \mathcal{X} : (x_1, \ldots, x_s) \in \mathcal{A}\}), \quad \mathcal{A} \in \mathbb{R}^s.$$

(b) This is a Poisson process: any set $\mathcal{A}$ that does not intersect with $\mathcal{E}'$ has mean number of points $\mu(\mathcal{A})$, where $\mu(\mathcal{A}) = 0$, and any that does has mean number of points $\mu(\mathcal{A}) = |\mathcal{A} \cap \mathcal{E}'|$. Clearly the Poisson axioms are satisfied for disjoint sets that intersect with $\mathcal{E}'$.

The mapping theorem fails because the projection $(x, y) \mapsto y$ collapses $\mathcal{E}'$ to a point $(0, 0)$, and the probability that there are two or more points there is greater than $1 - e^{-t} - e^{-t}$ for any real t, so it must equal unity. Condition (i) of the theorem fails.

(c) Consider the function

$$g_1(t, x) = t, \quad t \in [0, 1], x \in (0, \infty).$$

Let $0 \le t \le 1$. We have

$$\mu_1(\{t\}) = \mu\{g_1^{-1}(\{t\})\} = \mu\{\{t\} \times (0, \infty)\} = 0,$$

by the same argument as in (a); the space $\mathcal{X}$ is not compact here but the argument is the same. Consider now a compact $A = [t_1, t_2]$, where $0 \le t_1 < t_2 \le 1$. We have

$$\mu_1(A) = \mu\{g_1^{-1}(A)\} = \mu\{[t_1, t_2] \times (0, \infty)\} = (t_2 - t_1) \times 1 = (t_2 - t_1) < \infty.$$

Hence, applying the mapping theorem, we obtain that $\{T_j\}$ is a Poisson process on $[0, 1]$ with mean measure

$$\mu_1([t_1, t_2]) = t_2 - t_1, \quad 0 \le t_1 < t_2 \le 1.$$

Now consider the function

$$g_2(t, x) = x, \quad t \in [0, 1], x \in (0, \infty).$$

For $x \in (0, \infty)$, we have $\mu_2(x) = \mu\{g_2^{-1}(x)\} = \mu([0, 1] \times \{x\}) = 0$ by the same argument as before. Moreover, for a compact $B = [x_1, x_2]$, $0 < x_1 < x_2 < \infty$, $\mu_2(B) = \mu\{g_2^{-1}(B)\} = \mu([0, 1] \times B) < \infty$ (the last inequality can for instance be seen by integrating the intensity function). Finally, let $B = (x, \infty)$ for $x > 0$. We have that

$$\mu_2(B) = \mu\{g_2^{-1}(B)\} = \mu\{[0, 1] \times (x, \infty)\} = 1 \times (1 + \xi x)^{-1/\xi}.$$

Thus, applying the mapping theorem, we obtain that $\{X_j\}$ is a Poisson process with mean measure $\mu_2\{[x, \infty)\} = (1 + \xi x)^{-1/\xi}, x > 0$.

Solution 3

(a) The waiting time to the first event and the intervals between events all have $\exp(\lambda)$ distributions, so $\mathrm{E}(W_A) = \mathrm{E}(W_B) = 1/\lambda$, but

$$\mathrm{E}(1/W_A) = \mathrm{E}(1/W_B) = \int_0^\infty x^{-1}\, \lambda e^{-\lambda x}\, \mathrm{d}x = \infty :$$

these are poor estimators of λ.

(b) If $T_n > t$ then the number of events before t must be at most $n - 1$, i.e., $N(t) \le n - 1$. Since $N(t) \sim \text{Poiss}(\lambda t)$, we have

$$\mathrm{P}(T_n > t) = \sum_{r=0}^{n-1} \mathrm{P}\{N(t) = r\} = e^{-\lambda t} + \lambda t e^{-\lambda t} + (\lambda t)^2 e^{-\lambda t}/2! + \cdots + (\lambda t)^{n-1} e^{-\lambda t}/(n-1)!,$$

and thus $f_{T_n}(t) = -\mathrm{dP}(T_n > t)/\mathrm{d}t$ equals

$$\lambda e^{-\lambda t} - \lambda e^{-\lambda t} + \lambda^2 e^{-\lambda t} - \lambda^2 e^{-\lambda t} + \lambda^3 e^{-\lambda t}/2! - \cdots + \lambda^n t^{n-1} e^{-\lambda t}/n! = \lambda^n t^{n-1} e^{-\lambda t}/n!,$$

after most of the terms in the sum cancel.

(c) $W_C = W_B + X$, where X is the time from the event before t' to t'. This also has an $\exp(\lambda)$ distribution, independent of W_A, because there is no directionality in a Poisson process and events in separate intervals are independent. Hence $\mathrm{E}(W_C) = \mathrm{E}(W_B) + \mathrm{E}(X) = 2/\lambda$.

We see from (b) that W_C, which has the distribution of the waiting time to a second event, has density function

$$\lambda^2 t \exp(-\lambda t), \quad t > 0,$$

so

$$\mathrm{E}(W_C^{-1}) = \int_0^\infty t^{-1}\lambda^2 t \exp(-\lambda t)\,\mathrm{d}(t) = \lambda^2 \int_0^\infty t^{-1}\lambda^2 t \exp(-\lambda t)\,\mathrm{d}(t) = \lambda,$$

i.e., W_C^{-1} is an unbiased estimator for λ. As it has infinite variance (check!) it is not a good estimator, but it is unbiased.

Solution 4

(a) A ball of radius r centred at x is empty with probability $\mathrm{P}[N\{B_r(x)\} = 0] = \exp\{-\lambda|B_r(x)|\}$. The nearest event is at least r away iff $N\{B_r(x)\} = 0$, so the corresponding density function is

$$-\mathrm{dP}[N\{B_r(x)\} = 0]/\mathrm{d}r = \lambda \mathrm{d}|B_r(x)|/\mathrm{d}r \times \exp\{-\lambda|B_r(x)|\}, r > 0.$$

As events are independent, the void probability and the resulting density would be the same if x was an arbitrary point in space.

(b) We see from (a) that $K(r) = \lambda|B_r(x)|/\lambda = |B_r(x)|$, which equals πr^2 when $D = 2$. Hence $L(r) = r$, departures from which will suggest departures from the Poisson process. It is easier to judge departures from a line than from a quadratic.

(c) The data and simulations produce Figure 1.

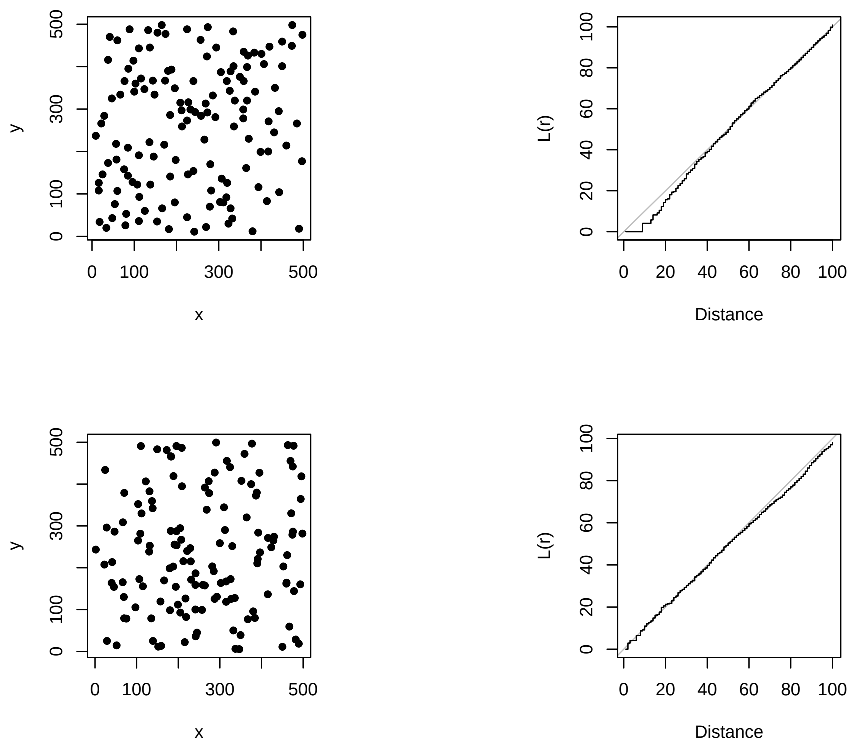

Figure 1: Example output for caveolae analysis and simulated data

The simulation works because the density function for a homogeneous Poisson process on $[0, 500]^2$ is uniform, which is also the case for the density generated by taking $500(U_1, U_2)$ with $U_1, U_2 \stackrel{\text{iid}}{\sim} U(0, 1)$.

The L-function for the data at the upper right panel seems to be too low for small r, which suggests that the caveolae tend not to be as close as the points of a Poisson process. This is confirmed by the lower panels (the simulations show more points close together than do the data) and reinforced by repeated simulations from the model.