Solution 1 Below we use the fact that the margins of the copula are uniform, and if $U \sim U(0,1)$, then $\mathrm{E}(U) = 1/2$ and $\mathrm{var}(U) = 1/12$.

(a) By the i.i.d. assumption, using that $\mathrm{P}(U_1 \geq V_1) = \mathrm{P}(U_1 \leq V_1) = 1/2$ we compute

$$\begin{aligned}
&\tau\mathrm{corr}(I\{U_1 > V_1\}, I\{U_2 > V_2\}) \\
&= \frac{\mathrm{E}(I\{U_1 > V_1\}I\{U_2 > V_2\}) - \mathrm{E}(I\{U_1 > V_1\})\mathrm{E}(I\{U_2 > V_2\})}{(\mathrm{E}(I\{U_1 > V_1\}) - \mathrm{E}(I\{U_1 > V_1\})^2)^{1/2}(\mathrm{E}(I\{U_2 > V_2\}) - \mathrm{E}(I\{U_2 > V_2\})^2)^{1/2}} \\
&= \frac{\mathrm{E}(I\{U_1 > V_1\}I\{U_2 > V_2\}) - (1/2)^2}{1/4} = 4\mathrm{E}(I\{U_1 > V_1\}I\{U_2 > V_2\}) - 1,
\end{aligned}$$

where $\mathrm{E}(I\{U_1 > V_1\}I\{U_2 > V_2\}) = \mathrm{E}(\mathrm{E}(I\{U_1 > V_1\}I\{U_2 > V_2\}|(U_1, U_2))) = E(C(U_1, U_2))$.

(b)

$$\rho = \mathrm{corr}(U_1, U_2) = \frac{\mathrm{E}(U_1U_2) - \mathrm{E}(U_1)\mathrm{E}(U_2)}{\mathrm{var}(U_1)^{1/2}\mathrm{var}(U_2)^{1/2}} = \frac{\mathrm{E}(U_1U_2) - \frac{1}{4}}{\frac{1}{12}} = 12\mathrm{E}(U_1U_2) - 3.$$

(c) When U_1 and U_2 are independent, we get a zero in the numerator in the second line of (a). Similarly in the first line in (b), $\mathrm{E}(U_1U_2) - 1/4 = \mathrm{E}(U_1)\mathrm{E}(U_2) - 1/4 = 0$.

Solution 2

(a) The copula is defined using $F_d^{-1}(u_d) = -1/\log u_d$ as

$$C(u_1, \ldots, u_D) = F\{F_1^{-1}(u_1), \ldots, F_1^{-1}(u_D)\} = \exp\{-V(-1/\log u_1, \ldots, -1/\log u_D)\}, \quad 0 < u_1, \ldots, < u_D.$$

(b) In terms of F we have $F^t(tz) = F(z)$, which when written as $F^t(z) = F(z/t)$ gives $C^t(u) = F^t(-1/\log u) = F(-1/t\log u) = F(-1/\log u^t) = C(u^t)$. Replacing u by $u^{1/t}$ gives the equation $C^t(u^{1/t}) = C(u)$, as required.

(c) It is easily checked that C_1 and C_2 are max-stable, but C_3 is not.

Solution 3

(a) Since the marginal distributions are unit Fréchet, we must have

$$G(z, \infty, \ldots, \infty) = \exp\{-V(z, \infty, \ldots, \infty)\} = \exp(-1/z), \quad z > 0,$$

i.e., $V(z, \infty, \ldots, \infty) = 1/z$, and this holds for any permutation of the arguments. For the max-stability we have

$$\mathrm{P}(Z_d \leq b_t + a_t z)^t = [\exp\{-1/(b + t + a_t z)\}]^t = \exp(-1/z)$$

if we set $b_t = 0$ and $a_t = t$, and this obviously holds for every $z, t > 0$.

(b) In the multivariate case we have $a_t = (a_t^1, \ldots, a_t^D)$ and $b_t = (b_t^1, \ldots, b_t^D)$, say. If

$$G(b_t^1 + a_t^1 z_1, \ldots, b_t^D + a_t^D z_D)^t = G(z_1, \ldots, z_D), \quad (z_1, \ldots, z_D) \in \mathcal{E}^*, t > 0,$$

then we see by replacing all but one of the zs by ∞ that we must have $a_t^d = t$ and $b_t^d = 0$ for each d; the marginal distributions must be max-stable with the same choice of as and bs. Therefore

$$\begin{aligned}
G(tz_1, \ldots, tz_D)^t &= \exp\{-tV(tz_1, \ldots, tz_D)\} \\
&= G(z_1, \ldots, z_D) = \exp\{-V(z_1, \ldots, z_D)\}, \quad (z_1, \ldots, z_D) \in \mathcal{E}^*, t > 0,
\end{aligned}$$

which gives the result.

For the second part, note that

$$\begin{aligned}\mathrm{P}\{\max(Z_1,\dots,Z_D)\le z\} &= \mathrm{P}(Z_1\le z,\dots,Z_D\le z)\\ &= \exp\{-V(z,\dots,z)\}\\ &= \exp\{-zV(z,\dots,z)/z\}\\ &= \exp\{-V(1,\dots,1)/z\}\\ &= \exp(-\theta_{\mathcal{D}}/z),\quad z>0.\end{aligned}$$

(c) Note that when $z\to\infty$, $\mathrm{P}(Z_1>z)=1-\exp(-1/z)=1/z+O(1/z^2)$ and

$$\begin{aligned}\mathrm{P}(Z_1>z,Z_2>z) &= 1-\mathrm{P}(Z_1\le z)-\mathrm{P}(Z_2\le z)+\mathrm{P}(Z_1\le z,Z_2\le z)\\ &= 1-2\exp(-1/z)+\exp\{-V(z,z)\}\\ &= 1-2(1-1/z)+\{1-V(1,1)/z\}+O(1/z^2)\\ &= (2-\theta)/z+O(1/z^2),\end{aligned}$$

so

$$\chi=\lim_{z\to\infty}\frac{\mathrm{P}(Z_1>z,Z_2>z)}{\mathrm{P}(Z_1>z)}=\lim_{z\to\infty}\frac{(2-\theta)/z+O(1/z^2)}{1/z+O(1/z^2)}=2-\theta.$$

In the two cases stated we have (independent variables), $\theta=V(1,1)=1/1+1/1=2$, so $\chi=0$, and (totally dependent variables) $\theta=V(1,1)=1/\min(1,1)=1$, so $\chi=1$. Thus θ can be seen as a measure of the asymptotic dependence between the two variables, with smaller values for higher dependence, and $\theta=D$ corresponding to independence.

Solution 4 Setting $z_1=\infty$ shows that the margin for z_2 is unit Fréchet, and the result for z_1 follows by symmetry. The density function can be written in the form $(z_1z_2)^{-2}V(z_1,z_2)\exp\{-V(z_1,z_2)\}$, which is positive, so this is a joint density. As $t\to\infty$,

$$F(tz_1,tz_2)^t=\exp\left[-\left\{z_1^{-1}+z_2^{-1}+(tz_1z_2)^{-1}\right\}\right]\to\exp\left\{-\left(z_1^{-1}+z_2^{-1}\right)\right\},$$

so it is not max-stable, and the limiting distribution for rescaled maxima is independence. For it to correspond to a Poisson process intensity, V should correspond to a measure on $\mathcal{E}^*$, so the measure of any rectangle $(z_1',z_1)\times(z_2',z_2)$ must be positive, i.e.,

$$V(z_1,z_2')+V(z_1',z_2)-V(z_1,z_2)-V(z_1',z_2')>0,\quad 0<z_1'<z_1, 0<z_2'<z_2.$$

However this equals $-(1/z_1'-1/z_1)(1/z_2'-1/z_2)$, which is negative, so there is no such Poisson process.