Problem 1 Suppose that a realisation of a Poisson process $\mathcal{P}$ on $\mathcal{E}$ with intensity function $\dot{\mu}(x)$ has been observed and its points are independently coloured red with probability $\gamma(x)$ or green with probability $1-\gamma(x)$, where $0 \le \gamma(x) \le 1$. This results in processes $\mathcal{P}_1 = \sum_j \delta_{X_j} I_{X_j}$ and $\mathcal{P}_2 = \sum_j \delta_{X_j}(1 - I_{X_j})$, where the I_{X_j} are independent Bernoulli variables with probabilities $\gamma(X_j)$ and δ_x is a unit point mass at x.

Show that the joint Laplace functional for two compactly-supported non-negative continuous functions f_1 and f_2 is

$$\mathrm{E}\left\{\exp\left(-\int f_1 \mathrm{d}\mathcal{P}_1 - \int f_2 \mathrm{d}\mathcal{P}_2\right)\right\} = \exp\left[-\int_{\mathcal{E}}\left\{1 - e^{-f_1(x)}\right\}\gamma(x)\dot{\mu}(x)\mathrm{d}x - \int_{\mathcal{E}}\left\{1 - e^{-f_2(x)}\right\}\{1-\gamma(x)\}\dot{\mu}(x)\mathrm{d}x\right],$$

and hence deduce that $\mathcal{P}_1$ and $\mathcal{P}_2$ are independent Poisson processes. Deduce that thinning a Poisson process by independently erasing events results in another Poisson process.

Hint: first condition on $\mathcal{P}$ and use independence of the $I(x_j)$ at the resulting points $\{x_1, \ldots, x_n\}$; then condition on the number of events in a compact set $\mathcal{A}$ containing the support $f_1 + f_2$ and use the argument of Lemma 7.

Problem 2 Let $\mathcal{P}$ be a Poisson process on the compact set $\mathcal{E} \subset \mathbb{R}^d$ with mean measure μ and intensity $\dot{\mu}$. Consider a mapping $g : \mathbb{R}^d \to \mathbb{R}^s$ that neglects coordinates of x, e.g., $g(x_1, \ldots, x_d) = (x_1, \ldots, x_s)$, for $s < d$.

(a) Show that $g(\mathcal{P})$ is a Poisson process and find its mean measure.

(b) Is a homogeneous Poisson process $\mathcal{P}$ on $\mathcal{E}' = \mathbb{R}_+$ also a Poisson process on $\mathcal{E} = \mathbb{R}^2$? Does the mapping theorem apply? If not, which condition fails?

(c) Let $\mathcal{P}$ consist of points $\{(T_j, X_j)\}$ on $\mathcal{X} = [0,1] \times (0, \infty)$ and that $\mu\{[t_1, t_2] \times [x, \infty)\} = (t_2 - t_1)(1 + \xi x)_+^{-1/\xi}$, for $0 \le t_1 < t_2 \le 1$ and $x > 0$. Are the processes $\{T_j\}$ and $\{X_j\}$ Poisson?

Problem 3 Alice, Bob and Carol don't know much about statistics, but want to estimate the rate λ of a homogeneous Poisson process that started a long time ago and which they begin to watch at time t'. Alice decides to record the length W_A of the interval after the first event following t'. Bob decides to record the time W_B from t' to the next event. Carol decides to record the length of the interval W_C surrounding t' (i.e., she waits to the next event, but also checks when the last event before t' occurred).

(a) Show that $\mathrm{E}(W_A) = \mathrm{E}(W_B)$ and these allow $1/\lambda$ to be estimated, but that $1/W_A$ and $1/W_B$ are poor estimators of λ.

(b) Show that the waiting time T_n to the nth event in a homogeneous Poisson process satisfies $T_n > t$ iff $\mathrm{P}\{N(t) \le n-1\}$, and deduce that

$$f_{T_n}(t) = \frac{\lambda^n t^{n-1}}{n!} e^{-\lambda t}, \quad t > 0.$$

(c) Show that $\mathrm{E}(W_C) = 2/\lambda$, but that $1/W_C$ is an unbiased estimator of λ.

Problem 4 Consider a homogeneous Poisson process of rate λ in $\mathbb{R}^D$.

(a) Show that the void probability of a ball $B_r(x)$ of radius r around an event at x is $\exp\{-\lambda|B_r(x)|\}$, where $|\cdot|$ denotes volume, and hence find the density function of the distance to the event nearest to x. Would this be the same if there was no event at x?

(b) Ripley's K-function is defined as

$$K(r) = \lambda^{-1}\mathrm{E}\left(\#\{\text{ number of events within distance } r \text{ of an arbitrary event}\}\right), \quad r > 0.$$

Find this function when $D = 2$, and explain why it might be preferable to plot $L(r) = \{K(r)/\pi\}^{1/2}$.

(c) The following code obtains and plots some data on the positions of $n = 138$ caveolae in a 500×500 unit square of muscle fibre, and then computes the estimate of $L(r)$. In practice it is important to allow for edge effects, but although the estimate is modified to do so, we do not discuss this here.

```
library(boot)
library(spatial)
data(cav)
par(pty="s",mfrow=c(2,2)) # square panels for plots
plot(cav,pch=16)
ppregion(xl=0,xu=500,yl=0,yu=500)
plot(Kfn(cav, fs=100), type="s", xlab="Distance", ylab="L(r)",
      panel.first=abline(0,1,col="grey"))
```

Do you think that the data could be a realisation of a Poisson process? Explain your reasoning.

To compare the data with simulations from a homogeneous binomial process with n events, we use

```
sim <- cav
sim$x <- 500*runif(138)
sim$y <- 500*runif(138)
plot(sim,pch=16)
plot(Kfn(sim, fs=100), type="s", xlab="Distance", ylab="L(r)",
      panel.first=abline(0,1,col="grey"))
```

Explain why this generates from the stated process. Does the simulated L-function shed light on the data? Compute the K-functions for some more sets of simulated data, then briefly summarise your conclusions.