Problem 1

(a) The bivariate Hüsler–Reiss model depends on a parameter $\lambda > 0$ and has exponent function

$$V(z_1, z_2) = \frac{1}{z_1}\Phi\left\{\frac{\lambda}{2} + \lambda^{-1}\log\left(\frac{z_2}{z_1}\right)\right\} + \frac{1}{z_2}\Phi\left\{\frac{\lambda}{2} + \lambda^{-1}\log\left(\frac{z_1}{z_2}\right)\right\},$$

where Φ denotes the standard normal distribution function. Show that letting $\lambda \to \infty$ and 0 yields independence and total dependence. Find $\theta = V(1, 1)$ for this model. and give its range of values.

(b) The bivariate negative logistic distribution has exponent function

$$V(z_1, z_2) = 1/z_1 + 1/z_2 - (z_1^\alpha + z_2^\alpha)^{-1/\alpha}, \quad \alpha > 0.$$

Show that letting $\alpha \to \infty$ and 0 yields total dependence and independence. Find $\theta = V(1, 1)$ for this model and give its range of values.

Problem 2 Let Z_1, Z_2 be independent unit Fréchet variables, and define $X_1 = Z_1$ and $X_2 = \max(aZ_1, Z_2)/(a + 1)$, where $a \geq 0$. Show that the exponent function for X_1, X_2 is

$$V(z_1, z_2) = \max\left\{\frac{1}{z_1}, \frac{a}{(a+1)z_2}\right\} + \frac{1}{(a+1)z_2}, \quad z_1, z_2 > 0.$$

Find the corresponding Pickands function and hence show that the corresponding angular variable W takes the values $a/(2a + 1)$ and 1 with probabilities $(2a + 1)/\{2(a + 1)\}$ and $1/\{2(a + 1)\}$ respectively. Sketch samples from this distribution.

Problem 3 The joint cumulative distribution function of random variables $Y_1, \ldots, Y_D$ is of the form

$$\mathrm{P}(Y_1 \leq y_1, \ldots, Y_D \leq y_D) = \exp\left[-V\{z_1(y_1), \ldots, z_D(y_D)\}\right],$$

where the function V possesses all its derivatives and the functions z_d are differentiable. Both V and the z_d depend on parameters for which a likelihood is sought.

(a) Find the joint density of Y_1 and Y_2 when $D = 2$ and hence verify the formula on slide 195.

(b) Compute the joint density for $D = 3$. What computational problem do you foresee for $D = 10$, say?

Problem 4 To run the following exercises first load the packages:

```
load(evd, texmex, mev, ismev)
```

The purpose of this problem sheet is to familiarise you with some of the R tools for modelling bivariate extremes. We shall work with a dataset from the `texmex` package that contains daily maxima of measurements of five air pollutants at Leeds, UK. The `summer` data contains observations from April to July, and the `winter` data has observations from November to February. For the purpose of the exercise, we shall assume that there are no issues with seasonality or non-stationarity. You can load the data using

```
data(summer) # to load the summer data, or
data(winter) # to load the winter data
```

We focus on modelling bivariate extremal dependence between two of the pollutants, NO2 and PM10, during the summer, which you may select via using the code

```
NO2 <- summer[,2]; PM10 <- summer[,5]
```

A preliminary step is to plot the data; see Figure 1.

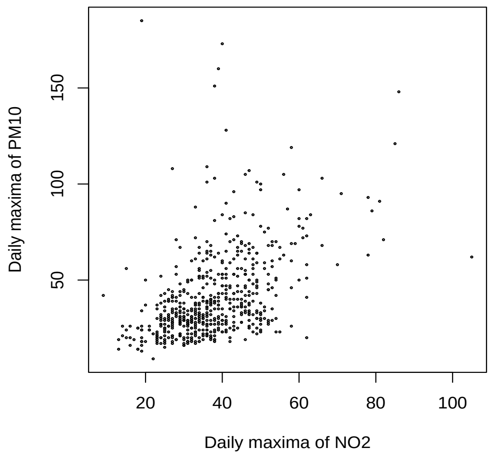

Figure 1: Daily maxima of NO2 and PM10.

(a) We first study the extremes of daily maxima by fitting the bivariate logistic extremal dependence model using the code

```
(ModelD1 <- fbvevd(cbind(NO2, PM10), model="log"))
```

To investigate the fit of the model you can use

```
# marginal fits
par(mfrow=c(1,4)); plot(ModelD1, mar=1)
plot(ModelD1, mar=2)

# bivariate fit
par(mfrow=c(2,3)); plot(ModelD1)
```

Do the diagnostic plots seem reasonable?

(b) Now we fit the asymmetric logistic dependence structure using the argument `model="alog"` in the `fbvevd` call. To compare the fits of the models (the logistic model is nested within the asymmetric logistic, which has two additional parameters), we run the code

```
cp <- ModelD1$dev-ModelD2$dev
qchisq(cp, df=2)
```

What do you conclude about the presence of asymmetry?

(c) Try modelling the bivariate extremal dependence with other models, such as the Hüsler–Reiss, negative logistic or bilogistic. You can use the `?fbvevd` command to find the arguments you need. Inspect the fits. Does one particular dependence structure perform appreciably better than others? You might compare them using the AIC (slide 213 of the lecture notes.)

(d) In this part we work with weekly maxima of NO2 and PM10, obtained using the code

```
#  To split 578 observations into blocks of 7 we pad the data with zeros
NO2[578:581] <- 0; PM10[578:581] <- 0;

NO2_w <- apply(matrix(NO2, ncol=7, byrow=F), 1, function(x)max(x))
PM10_w <- apply(matrix(PM10, ncol=7, byrow=F), 1, function(x)max(x))
```

Try to fit a bivariate extremal dependence model similar to parts (a) and (b). Discuss whether you notice any differences in the marginal and bivariate fits and in the strength of extremal dependence.

(e) Consider again the air pollutants NO2 and PM10 from the previous exercise. We will study extremal behaviour in the framework of multivariate peaks over thresholds.

(a) Try to fit bivariate extremal dependence models to the exceedances of NO2 and PM10. Start with the simple logistic dependence structure. You may set the marginal thresholds to correspond to the 95-th percentiles, using the code

```
airpollutants <- cbind(NO2, PM10)
(POT1  <-  evd::fbvpot(airpollutants, apply(airpollutants, 2, quantile, 0.95),
model="log"))
```

Inspect the resulting marginal and bivariate diagnostic plots; do you notice any differences between the estimated parameters and the strength of extremal dependence and those in the first exercise?

Problem 5 Consider again the air pollutants NO2 and PM10 from the previous exercise. We will study extremal behaviour in the framework of multivariate peaks over thresholds.

(a) Try fitting bivariate extremal dependence models to the exceedances of NO2 and PM10. Start with the logistic dependence model with marginal thresholds at the 0.95 quantiles, using the code

```
library(evd)
airpollutants<-cbind(NO2, PM10)
(POT1 <- fbvpot(airpollutants, apply(airpollutants, 2, quantile, 0.95), model="log"))
```

Do the fits and resulting marginal and bivariate diagnostic plots show major differences between such models and the fits for the previous exercise?

(b) Try to model bivariate extremal dependence via the asymmetric logistic dependence structure, i.e., setting the argument `model="alog"`, and inspect the resulting fit. Do your conclusions regarding asymmetry differ from those of part (b) in the previous exercise?