Problem 1 Let the continuous random variables (U_1, U_2) and (V_1, V_2) be independent and identically distributed with copula $C(u_1, u_2)$.

(a) Show that Kendall's tau may be written $\tau = \text{corr}\{I(U_1 > V_1), I(U_2 > V_2)\} = 4\text{E}\{C(U_1, U_2)\} - 1$.

(b) Spearman's $\rho = \text{corr}(U_1, U_2)$ is another measure of concordance. Show that $\rho = 12\text{E}(U_1 U_2) - 3$.

(c) Show that if U_1 and U_2 are independent, then $\tau = \rho = 0$.

Problem 2 Let $Z = (Z_1, \dots, Z_D)$ have joint distribution function

$$F(z_1, \dots, z_D) = \exp\{-V(z_1, \dots, z_D)\}, \quad z_1, \dots, z_D > 0,$$

where, writing $z = (z_1, \dots, z_D)$ for brevity, $tV(tz) = V(z)$ for all $z > 0$ and the Z_d have marginal unit Fréchet distributions.

(a) Show that the copula corresponding to F is

$$C(u_1, \dots, u_D) = \exp\{-V(-1/\log u_1, \dots, -1/\log u_D)\}, \quad 0 < u_1, \dots, < u_D.$$

(b) Writing $u = (u_1, \dots, u_D)$ for brevity, show that the max-stability of F leads to the property $C(u^{1/t})^t = C(u)$. Such a copula is said to be *max-stable*.

(c) If $u = (u_1, \dots, u_D)$, $0 < \alpha < 1$ and $\rho > 0$, are the following copulas max-stable:

$$C_1(u) = \exp\left[-\left\{\sum_{d=1}^{D}(-\log u_d)^{1/\alpha}\right\}^{\alpha}\right], \quad C_2(u) = \exp\left[\sum_{d=1}^{D}\log u_d - \left\{\sum_{d=1}^{D}(-\log u_d)^{-\rho}\right\}^{-1/\rho}\right],$$

$$C_3(u) = \exp\left\{\sum_{d=1}^{D}\log u_d - \prod_{d=1}^{D}(-\log u_d)\right\}?$$

Problem 3 The joint distribution function of $Z_1, \dots, Z_D$ is of the form

$$G(z_1, \dots, z_D) = \exp\{-V(z_1, \dots, z_D)\}, \quad (z_1, \dots, z_D) \in \mathcal{E}^* = [0, \infty)^D - \{(0, \dots, 0)\},$$

and has unit Fréchet marginal distributions, i.e., $\text{P}(Z_d \le z_d) = \exp(-1/z_d)$ for each $d \in \mathcal{D} = \{1, \dots, D\}$.

(a) Show that $V(z, \infty, \dots, \infty) = 1/z$ for every permutation of the arguments of V, and that the unit Fréchet distribution is max-stable, i.e., there exist $a_t > 0$ and $b_t \in \mathbb{R}$ such that $\text{P}(Z \le b_t + a_t z)^t = \text{P}(Z \le z)$ for every $z, t > 0$.

(b) If the joint distribution G^* of $Z_1, \dots, Z_D$ is max-stable, i.e., $\{G^*(tz)\}^t = G^*(z)$ for all t and z, show that $V(z_1, \dots, z_D) = tV(tz_1, \dots, tz_D)$ for every $t > 0$ and $(z_1, \dots, z_D) \in \mathcal{E}^*$; V is said to be *homogeneous of order* -1. Deduce that $V(z, \dots, z) = V(1, \dots, 1)/z = \theta_{\mathcal{D}}/z$, say, and hence show that $\text{P}\{\max(Z_1, \dots, Z_D) \le z\} = \exp(-\theta_{\mathcal{D}}/z)$, a Fréchet distribution with parameter $\theta_{\mathcal{D}}$.

(c) Let $\theta = V(1, 1)$ when $D = 2$. Show that $\chi = \lim_{z\to\infty} \text{P}(Z_2 > z \mid Z_1 > z) = 2 - \theta$. Find θ when $V(z_1, z_2) = 1/z_1 + 1/z_2$ and $V(z_1, z_2) = 1/\min(z_1, z_2)$, and hence interpret it in terms of the degree of dependence between Z_1 and Z_2.

Problem 4 Show that the distribution

$$F(z_1, z_2) = \exp\left[-\left\{z_1^{-1} + z_2^{-1} + (z_1 z_2)^{-1}\right\}\right], \quad z_1, z_2 > 0,$$

has unit Fréchet margins. Is it max-stable? If not, what is the limiting distribution of appropriately rescaled componentwise maxima from F? Does this V correspond to the measure of a Poisson process?