Risk and Environmental Sustainability: Dummy Examination

11 March 2011

Instructions: The time allotted for the examination is 180 minutes. You may answer in either English or French. No written material may be brought into the examination. Full marks may be obtained with complete answers to four questions. The final mark will be based on the best four solutions.

First name:

Last name:

SCIPER number:

Exercise	Marks	Indicative marks
1		10
2		10
3		10
4		10
5		10
Total:		40

Question 1

(a) Define the *Poisson process* and state the *mapping theorem.*

(b) A Poisson process $\mathcal{P} = \{(r_j, w_j) : j = 1, \ldots\}$ with state space $\mathcal{E} = (0, \infty) \times \mathbb{R}_+^D$ has intensity function $\dot{\mu}(r, w) = r^{-2} \times \dot{\nu}(w)$, where $\dot{\nu}$ is a probability density function on $\mathbb{R}_+^D$, i.e., $w = (w_1, \ldots, w_D)$, and $\int w_d\, \dot{\nu}(w)\, \mathrm{d}w = 1$ for $d = 1, \ldots, D$. Define the transformation $g : \mathcal{E} \to \mathcal{E}^*$ given by $q = g(r, w) = rw = (rw_1, \ldots, rw_D)$ and let $\mathcal{P}^* = \{q_j : j = 1, \ldots\} \subset \mathcal{E}^* = \mathbb{R}_+^D$. Verify that the mapping theorem applies to g and deduce that $\mathcal{P}^*$ is a Poisson process on $\mathcal{E}^*$ with mean measure

$$\mu^*(\mathcal{A}^*) = \mu\{g^{-1}(\mathcal{A}^*)\}, \quad \mathcal{A}^* \subset \mathcal{E}^*.$$

(c) Show that the intensity of $\mathcal{P}^*$ can be written as

$$\dot{\mu}^*(q) = \int_0^\infty u^D \dot{\nu}(uq)\, \mathrm{d}u, \quad q \in \mathcal{E}^*.$$

Question 2

(a) State the *extremal types theorem* and outline its importance. Explain what is meant by the terms *max-stability* and *generalized extreme-value (GEV) distribution.*

(b) If $X_1, \ldots, X_m \stackrel{\text{iid}}{\sim} G$, where G is the GEV distribution with parameters η, τ and ξ, find the distribution of $M = \max(X_1, \ldots, X_m)$ and give its parameters.

(c) The GEV distribution is fitted to annual maxima of daily sea levels at Fremantle, South Australia. The plots below are produced by the `R` function `gev.diag`. Discuss the adequacy of the fitted model and say what they tell you about the sign of the shape parameter.

(d) Use the output below to construct a 95% confidence interval for the shape parameter. What do the shape parameter estimate and its confidence interval tell you about the tail of the fitted distribution? Does your conclusion seem reasonable in the light of your general knowledge?

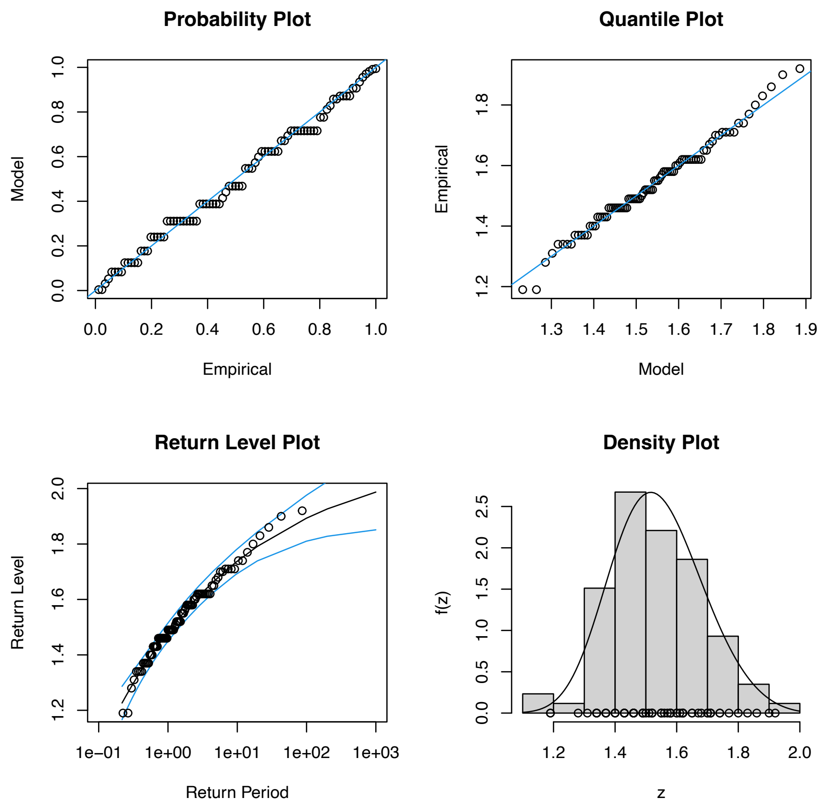
```
(fremantle.fit <- gev.fit(fremantle))

$conv
[1] 0

$nllh
[1] -43.56663

$mle
[1]  1.4823409  0.1412671 -0.2174320

$se
[1] 0.01672502 0.01149461 0.06377394
```



Question 3 The distribution function F of the continuous scalar random variable X satisfies

$$F(b_n + a_n x)^n \to \exp\left\{-(1+\xi x)_+^{-1/\xi}\right\}, \quad x, \xi \in \mathbb{R}, \quad n \to \infty,$$

where $a_+ = \max(a, 0)$ and $\{b_n\} \subset \mathbb{R}$ and $\{a_n\} \subset \mathbb{R}_+$ are sequences of constants.

(a) Show that

$$\mathrm{P}\{X > b_n + a_n(x+u) \mid X > b_n + a_n u\} \to (1 + \xi x/\sigma)_+^{-1/\xi}, \quad x > 0, u \in \mathbb{R}, \quad n \to \infty,$$

where σ is to be determined, and interpret this limit in terms of the properties of X.

(b) If $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} F$ and $I(\cdot)$ denotes an indicator function, show that as $n \to \infty$, the random variable

$$N_{u,n} = \sum_{j=1}^{n} I(X_j > b_n + a_n u), \quad u \in \mathbb{R},$$

has a limiting Poisson distribution and find its mean.

(c) How are the results in (a) and (b) useful in the statistical analysis of extreme values?

(d) If the random variables $Z_1, Z_2, \ldots$ are independent and have the limiting distribution given by (a), and N_u has the limiting Poisson distribution in (b), find the distribution of $\max(Z_1, \ldots, Z_{N_u})$. How is this related to the generalized extreme-value distribution?

Question 4 Write an essay on the effects of dependence on the extremes of stationary time series, including discussions of the distribution of block maxima and of threshold exceedances, of the effect of dependence on return levels, and of related statistical procedures.

Question 5

(a) Let $G(z_1, z_2)$ be a bivariate extreme value distribution with unit Fréchet marginal distributions. Derive the key properties of the corresponding exponent function.

(b) Show that if $(Z_1, Z_2) \sim G$, then $M = \max(Z_1, Z_2)$ has a Fréchet distribution and give its parameter. By considering the cases

$$V(z_1, z_2) = 1/z_1 + 1/z_2, \quad V(z_1, z_2) = 1/\min(z_1, z_2),$$

or otherwise, give an interpretation of $\theta = V(1, 1)$.

(c) Show that if X is a continuous random variable with distribution function $F(x) = e^{-1/x}$, for $x > 0$, then $F(X) \sim U(0, 1)$.

(d) Given that

$$|x - y| = 2\max(x, y) - (x + y), \quad x, y \in \mathbb{R},$$

derive the expectation of

$$T = |F(Z_1) - F(Z_2)|$$

and explain how you might use this result to estimate $\theta = V(1, 1)$ from data.