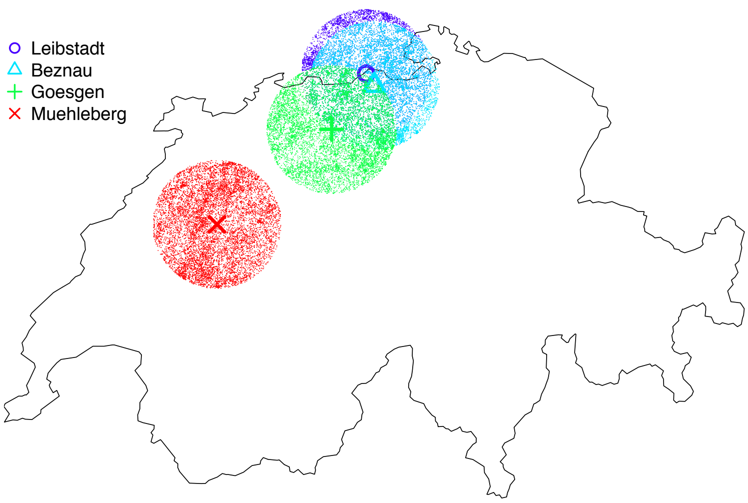
2 Poisson Process slide 42

2.1 Point Processes slide 43

Lightning strikes within 30km of the Swiss nuclear sites



Point process

- A **point process** is a stochastic model for a **point pattern** $\mathcal{P} = \{x_1, x_2, \ldots\}$ lying in a **state space** $\mathcal{E}$. We also call a point an **event**.
- We visualise $\mathcal{E} \subset \mathbb{R}^2$, but $\mathcal{E}$ might be more complex, e.g., $\mathcal{E} = \mathbb{R} \times \mathcal{C}$, where $\mathcal{C}$ is a space of functions—then a 'point' would be $x = (u, f) \in \mathcal{E}$, with $u \in \mathbb{R}$ and $f \in \mathcal{C}$.
- The set $\mathcal{E}$ must allow us to count how many points of $\mathcal{P}$ lie in any suitable subset $\mathcal{A} \subset \mathcal{E}$, giving

$$N(\mathcal{A}) = |\mathcal{P} \cap \mathcal{A}| = \sum_x I(x \in \mathcal{P} \cap \mathcal{A}), \quad \mathcal{A} \subset \mathcal{E},$$

  where $I(\cdot)$ is an indicator function.
- Two points cannot exactly coincide: $\mathcal{P}$ must be **simple** (or **orderly**) — otherwise we would not know how many points there are.
- If you know about measures . . . the function $N(\mathcal{A})$ is
  - a **counting measure** on $\mathcal{E}$, since it counts the number of elements of $\mathcal{P}$ in any (measurable) set $\mathcal{A}$,
  - a **Radon measure** if $N(\mathcal{A}) < \infty$ for any $\mathcal{A}$ compact (in a suitable topology on $\mathcal{E}$),
  - a **random measure** if the points $\mathcal{P}$ arise at random, since then $N(\mathcal{A})$ is a random variable computed from the (random) $\mathcal{P}$.

Laplace transform

- If it exists, the **Laplace transform** of a scalar random variable X is defined as

$$\mathrm{E}\left\{\exp(-tX)\right\} = M_X(-t),$$

where M_X is the **moment-generating function (MGF)**. This is useful because
  - there is a bijection between distributions and MGFs, i.e., if we recognise M_X, then we know the corresponding distribution;
  - the **continuity theorem** tells us that if $\{X_n\}$, X have CDFs $\{F_n\}$, F for which the MGFs $M_n(t)$, $M(t)$ exist and there exists $a > 0$ such that

$$\lim_{n\to\infty} M_n(t) = M(t), \quad 0 \le |t| < a,$$

then $X_n \xrightarrow{D} X$, i.e., X_n converges in distribution (weakly, in law) to X.
  - Hence for large enough n we can approximate the distribution of X_n by that of X.
- On the next slide we will extend this to point processes, but first, a simple example:

Theorem 1 (Law of small numbers) *If $X_n \sim B(n, p_n)$ and $np_n \to \lambda > 0$ when $n \to \infty$, then the limiting distribution of X_n is $\mathrm{Pois}(\lambda)$, i.e., $X_n \xrightarrow{D} X$, where $X \sim \mathrm{Pois}(\lambda)$.*

Note to Theorem 1

- The MGF of X is

$$M(t) = \mathrm{E}(e^{tX}) = \sum_{x=0}^{\infty} e^{tx}\lambda^x e^{-\lambda}/x! = e^{-\lambda}\sum_{x=0}^{\infty}(\lambda e^t)^x/x! = \exp\left\{\lambda(e^t-1)\right\}, \quad t \in \mathbb{R}.$$

- The MGF of X_n is

$$M_n(t) = \mathrm{E}(e^{tX_n}) = \sum_{x=0}^{n} e^{tx}\binom{n}{x}p_n^x(1-p_n)^{n-x} = \left(1-p_n+p_n e^t\right)^n, \quad t \in \mathbb{R}.$$

Let $p_n = \lambda_n/n$, where $\lambda_n \to \lambda$, and note that as $n \to \infty$ and for any real t,

$$\left(1-p_n+p_n e^t\right)^n = \left(1 + \frac{\lambda_n(e^t-1)}{n}\right)^n \to \exp\left\{\lambda(e^t-1)\right\}.$$

- As $M_n(t) \to M(t)$ for all real t, the continuity theorem implies that $X_n \xrightarrow{D} X$.

Laplace functional

- We specify properties of $\mathcal{P}$ through the finite-dimensional distributions of $N(\cdot)$, i.e.,

$$\mathrm{P}\{N(\mathcal{A}_1) = n_1, \ldots, N(\mathcal{A}_k) = n_k\}, \quad n_1, \ldots, n_k \in \{0, 1, 2, \ldots, \},$$

  for all possible choices of sets $\mathcal{A}_1, \ldots, \mathcal{A}_k$, and all $k = 0, 1, 2, \ldots$.
- An efficient way to do this is through the **Laplace functional**,

$$\mathcal{L}_{\mathcal{P}}(f) = \mathrm{E}\left\{\exp\left(-\int f \mathrm{d}\mathcal{P}\right)\right\}, \quad \text{where} \quad \int f \mathrm{d}\mathcal{P} = \int f(x)\mathcal{P}(\mathrm{d}x) = \sum_{x \in \mathcal{P}} f(x),$$

  for functions $f \geq 0$ that are positive only on a bounded set. If $f(x) = \sum_r t_r I(x \in \mathcal{A}_r)$, then $\mathcal{L}_{\mathcal{P}}(f)$ is the joint MGF for the $N(\mathcal{A}_r)$.
- Under mild conditions, there is
  - a bijection between point processes and Laplace functionals; and
  - the continuity theorem can be generalised.

Convergence of point processes

Definition 2 *A sequence of random variables $\{X_n\}$ with corresponding CDFs $\{F_n\}$* **converges weakly (or in distribution)** *to a random variable X with CDF F if*

$$\lim_{n \to \infty} F_n(x) = F(x) \text{ at every } x \text{ where } F \text{ is continuous.}$$

Definition 3 *A sequence of point processes $\{\mathcal{P}_n\}$ with corresponding counts $\{N_n(\cdot)\}$ on $\mathcal{E}$* **converges weakly (or in distribution)** *to a point process $\mathcal{P}$ with count $N(\cdot)$, written $\mathcal{P}_n \xrightarrow{D} \mathcal{P}$, if for all choices of k and all compact sets $\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathcal{E}$ such that*

$$\mathrm{P}\{N(\partial\mathcal{A}_j) = 0\} = 1, \quad j = 1, \ldots, k,$$

where $\partial\mathcal{A}_j$ is the boundary of $\mathcal{A}_j$,

$$\{N_n(\mathcal{A}_1), \ldots, N_n(\mathcal{A}_k)\} \xrightarrow{D} \{N(\mathcal{A}_1), \ldots, N(\mathcal{A}_k)\}, \quad n \to \infty.$$

Theorem 4 (No proof) *The point processes $\mathcal{P}_1, \mathcal{P}_2, \ldots$ converge weakly to the point process $\mathcal{P}$ on $\mathcal{E}$ if and only if the corresponding Laplace functionals converge for every continuous non-negative function f on $\mathcal{E}$ with compact support, i.e., as $n \to \infty$,*

$$\mathcal{L}_{\mathcal{P}_n}(f) = \mathrm{E}\left\{\exp\left(-\int f \mathrm{d}\mathcal{P}_n\right)\right\} \to \mathcal{L}_{\mathcal{P}}(f) = \mathrm{E}\left\{\exp\left(-\int f \mathrm{d}\mathcal{P}\right)\right\}.$$

Kallenberg's theorem

- **Kallenberg's theorem** gives another way to establish the weak convergence of $\{\mathcal{P}_n\}$ to a simple process $\mathcal{P}$ when $\mathcal{E} \subset \mathbb{R}$.
- For any $\mathcal{A} \subset \mathcal{E}$, let $N_n(\mathcal{A}) = |\mathcal{P}_n \cap \mathcal{A}|$. Then if
  - $\mathcal{B} \subset \mathcal{E}$ is any interval,
  - $\mathcal{C}$ is any finite union of disjoint sub-intervals of $\mathcal{E}$,

  and if

$$\mathrm{E}\{N_n(\mathcal{B})\} \to \mathrm{E}\{N(\mathcal{B})\}, \quad \mathrm{P}\{N_n(\mathcal{C}) = 0\} \to \mathrm{P}\{N(\mathcal{C}) = 0\}, \quad n \to \infty, \qquad (2)$$

  then $\mathcal{P}_n$ converges weakly to $\mathcal{P}$.
- When $\mathcal{E} \subset \mathbb{R}^D$, the same result holds if intervals are replaced by **rectangles**,

$$(a, b] = \{x = (x_1, \dots, x_D) : a_d < x_d \le b_d, d = 1, \dots, D\} \subset \mathcal{E},$$

  where $a_d < b_d$ for each d.
- Thus weak convergence of point processes to a simple limiting process in $\mathbb{R}^D$ entails establishing convergence of expected counts for rectangles and of the **void probabilities** of finite unions of rectangles.
- See Kingman (1993) *Poisson Processes* and Daley and Vere-Jones (2002, 2008), *An Introduction to the Theory of Point Processes*.

2.2 Poisson Processes

Poisson process

Definition 5 *A* **Poisson process** *is a random countable subset $\mathcal{P}$ of a state space $\mathcal{E}$ such that*

- *the random variables $N(\mathcal{A}_1), \dots, N(\mathcal{A}_k)$ corresponding to any collection of disjoint subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$ of $\mathcal{E}$ are independent; and*
- *for any $\mathcal{A} \subset \mathcal{E}$, $N(\mathcal{A})$ has the Poisson distribution with mean $\mu(\mathcal{A})$, where $0 \le \mu(\mathcal{A}) \le \infty$, and $\mu(\mathcal{A}) < \infty$ for compact $\mathcal{A}$.*

Comments:

- if $\mathcal{A} = \bigcup_j \mathcal{A}_j$ is a countable union of disjoint sets, then $N(\mathcal{A}) = \sum_j N(\mathcal{A}_j)$, so $\mu(\mathcal{A}) = \sum_j \mu(\mathcal{A}_j)$, and μ is a measure; called the **mean measure** of $\mathcal{P}$;
- μ must be **diffuse**, i.e., $\mu(\{x\}) = 0$ for every $x \in \mathcal{E}$;
- if $\mathcal{E} \subset \mathbb{R}^D$, $\mathcal{A} = [a_1, x_1] \times \cdots \times [a_D, x_D]$, and if

$$\dot{\mu}(x_1, \dots, x_D) = \frac{\partial^D \mu(\mathcal{A})}{\partial x_1 \cdots \partial x_D}$$

  exists and is finite, then $\dot{\mu}$ is called the **intensity function** of $\mathcal{P}$;
- if $\dot{\mu}(x) \equiv \dot{\mu}$, then $\mathcal{P}$ is called **homogeneous**. Otherwise it is **inhomogeneous**.
- We simplify notation by replacing $\mu\{(a, b]\}$ by $\mu(a, b]$, etc.

Conditioning

Theorem 6 (Conditioning) *Let $\mathcal{P}$ be a Poisson process with mean measure μ, and suppose that $\mathcal{A} \subset \mathcal{E}$ is such that $0 < \mu(\mathcal{A}) < \infty$. Conditional on the event $N(\mathcal{A}) = n$, the n points of $\mathcal{P} \cap \mathcal{A}$ have the same distribution as n points generated independently at random in $\mathcal{A}$ with measure $\mu_{\mathcal{A}}(\mathcal{B}) = \mu(\mathcal{B})/\mu(\mathcal{A})$, for $\mathcal{B} \subset \mathcal{A}$.*

- ☐ If μ has intensity $\dot{\mu}(x)$, then we can generate points of $\mathcal{P}$ in $\mathcal{A}$ by
  - generating a value n of $N(\mathcal{A}) \sim \text{Poiss}\{\mu(\mathcal{A})\}$;
  - then generating $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \dot{\mu}(x)/\mu(\mathcal{A})$ for $x \in \mathcal{A}$.
- ☐ The process generated at the second step is a **binomial process**.

Lemma 7 *The* **Laplace functional** *of a Poisson process $\mathcal{P}$ on $\mathcal{E}$ with mean measure μ is*

$$\mathcal{L}_{\mathcal{P}}(f) = \exp\left[-\int_{\mathcal{E}} \left\{1 - e^{-f(x)}\right\} \mu(\mathrm{d}x)\right].$$

Note to Theorem 6

If we observe a Poisson process with intensity $\dot{\mu}(x)$ on the set $\mathcal{A}$, and there are points at $\{x_1, \ldots, x_n\}$, then the corresponding probability element is

$$\exp\{-\mu(\mathcal{A})\} \times \prod_{j=1}^{n} \dot{\mu}(x_j), \quad \{x_1, \ldots, x_n\} \subset \mathcal{A}.$$

Properties of the Poisson process imply that $N(\mathcal{A})$ has a Poisson distribution with mean $\mu(\mathcal{A})$, so the conditional density of the n points in $\mathcal{A}$, given that $N(\mathcal{A}) = n$, is the ratio

$$\frac{\exp\{-\mu(\mathcal{A})\} \times \prod_{j=1}^{n} \dot{\mu}(x_j)}{\mu(\mathcal{A})^n \exp\{-\mu(\mathcal{A})\}/n!} = n! \prod_{j=1}^{n} \left\{\frac{\dot{\mu}(x_j)}{\mu(\mathcal{A})}\right\}, \quad \{x_1, \ldots, x_n\} \subset \mathcal{A}.$$

Now consider the measure $\mu_{\mathcal{A}}(\mathcal{B}) = \mu(\mathcal{B})/\mu(\mathcal{A})$, for $\mathcal{B} \subset \mathcal{A}$, which is a probability measure on subsets of $\mathcal{A}$, because it is non-negative and $\mu_{\mathcal{A}}(\mathcal{A}) = 1$. The corresponding probability density is $\dot{\mu}(x)/\mu(\mathcal{A})$ ($x \in \mathcal{A}$), so the joint density for independent identically distributed variables $X_1, \ldots, X_n$ with distribution $\mu_{\mathcal{A}}$ is $\prod_{j=1}^{n}\{\dot{\mu}(x_j)/\mu(\mathcal{A})\}$, which is almost the conditional probability above. The additional factor $n!$ arises because the point process is unlabelled: the same density would arise for any of the $n!$ permutations of $X_1, \ldots, X_n$ that gave the outcome $\{x_1, \ldots, x_n\}$.

Note to Lemma 7

Let $f \geq 0$ have support only on a compact $\mathcal{A}$, so $\mu(\mathcal{A}) < \infty$. Conditional on $N(\mathcal{A}) = n$, $\int f(x)\mathcal{P}(\mathrm{d}x) = \sum_{j=1}^{n} f(X_j)$, where $\{X_1, \ldots, X_n\} \subset \mathcal{A}$ are independent with density $\dot{\mu}(x)/\mu(\mathcal{A})$. Thus

$$\begin{aligned} \mathrm{E}\left[\exp\left\{-\int f(x)\mathcal{P}(\mathrm{d}x)\right\} \,\middle|\, N(\mathcal{A}) = n\right] &= \mathrm{E}\left[\exp\left\{-\sum_{j=1}^{n} f(X_j)\right\} \,\middle|\, N(\mathcal{A}) = n\right] \\ &= \left\{\int_{\mathcal{A}} e^{-f(x)}\mu(\mathrm{d}x)/\mu(\mathcal{A})\right\}^n. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{E}\left[\exp\left\{-\int f(x)\mathcal{P}(\mathrm{d}x)\right\}\right] &= \sum_{n=0}^{\infty}\left\{\int_{\mathcal{A}} e^{-f(x)}\mu(\mathrm{d}x)/\mu(\mathcal{A})\right\}^n \frac{\mu(\mathcal{A})^n}{n!} e^{-\mu(\mathcal{A})} \\ &= \exp\left[\int_{\mathcal{A}} e^{-f(x)}\mu(\mathrm{d}x) - \mu(\mathcal{A})\right] \\ &= \exp\left[-\int_{\mathcal{A}}\left\{1 - e^{-f(x)}\right\}\mu(\mathrm{d}x)\right] \\ &= \exp\left[-\int_{\mathcal{E}}\left\{1 - e^{-f(x)}\right\}\mu(\mathrm{d}x)\right], \end{aligned}$$

as required, since $1 - \exp\{-f(x)\} \equiv 0$ outside $\mathcal{A}$.

Superposition and colouring

Theorem 8 (Superposition) *If $\mathcal{P}_1$, $\mathcal{P}_2$ are independent Poisson processes on $\mathbb{R}^D$ with mean measures μ_1, μ_2, then their union $\mathcal{P}_1 \cup \mathcal{P}_2$ is a Poisson process with mean measure $\mu_1 + \mu_2$.*

Theorem 8 extends to a countable number of Poisson processes.

Theorem 9 (Colouring) *Let $\mathcal{P}$ be a Poisson process with intensity $\dot{\mu}(x)$. Colour a point of $\mathcal{P}$ at x red with probability $\gamma(x)$; otherwise colour it green. Then the red and green sets of points $\mathcal{P}_{\mathrm{red}}$ and $\mathcal{P}_{\mathrm{green}}$ are independent Poisson processes with intensity functions*

$$\dot{\mu}_{\mathrm{red}}(x) = \dot{\mu}(x)\gamma(x), \quad \dot{\mu}_{\mathrm{green}}(x) = \dot{\mu}(x)\{1 - \gamma(x)\}.$$

The colouring theorem is in some sense the inverse of the superposition theorem, and it too applies with a countable number of colours.

Note to Theorem 8

☐ This looks easy using the Laplace functional for $\mathcal{P}_1 \cup \mathcal{P}_2$, which is

$$\mathcal{L}_{\mathcal{P}_1\cup\mathcal{P}_2}(f) = \mathrm{E}\left[\exp\left\{-\int f(x)(\mathcal{P}_1 \cup \mathcal{P}_2)(\mathrm{d}x)\right\}\right].$$

Now

$$\int f(x)(\mathcal{P}_1 \cup \mathcal{P}_2)(\mathrm{d}x) = \int f(x)\mathcal{P}_1(\mathrm{d}x) + \int f(x)\mathcal{P}_2(\mathrm{d}x),$$

and the two processes are independent, so

$$\begin{aligned} \mathrm{E}\left[\exp\left\{-\int f\mathrm{d}(\mathcal{P}_1 \cup \mathcal{P}_2)\right\}\right] &= \mathrm{E}\left\{\exp\left(-\int f\mathrm{d}\mathcal{P}_1\right)\right\} \times \mathrm{E}\left\{\exp\left(-\int f\mathrm{d}\mathcal{P}_2\right)\right\} \\ &= \exp\left\{-\int_{\mathcal{E}}\left(1-e^{-f}\right)\mathrm{d}\mu_1\right\} \times \exp\left\{-\int_{\mathcal{E}}\left(1-e^{-f}\right)\mathrm{d}\mu_2\right\} \\ &= \exp\left\{-\int_{\mathcal{E}}\left(1-e^{-f}\right)\mathrm{d}(\mu_1+\mu_2)\right\}, \end{aligned}$$

which is the Laplace functional of a Poisson process with mean measure $\mu_1 + \mu_2$.

☐ The catch with the argument above is the assumption that points of $\mathcal{P}_1$ and $\mathcal{P}_2$ do not coincide, so that

$$\mathrm{P}(\mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{A} = \emptyset) = 1$$

for any $\mathcal{A}$ for which $\mu_1(\mathcal{A})$, $\mu_2(\mathcal{A})$ are both finite. This is intuitively obvious but takes a bit of measure-theoretic work to prove.

Mapping

Theorem 10 (Mapping) *Let $\mathcal{P}$ be a Poisson process on $\mathcal{E}$ with mean measure μ, and suppose that the function $g : \mathcal{E} \to \mathcal{E}^*$ maps $\mathcal{E}$ into $\mathcal{E}^*$. Define*

$$\mu^*(\mathcal{A}^*) = \mu\{g^{-1}(\mathcal{A}^*)\}, \quad \mathcal{A}^* \subset \mathcal{E}^*.$$

If

(i) $\mu^*(\{x^*\}) = \mu^*(x^*) = 0$ for every $x^* \in \mathcal{E}^*$, and

(ii) $\mu^*(\mathcal{A}^*) < \infty$ for any compact $\mathcal{A}^*$,

then $\mathcal{P}^* = g(\mathcal{P})$ is a Poisson process on $\mathcal{E}^*$ with mean measure μ^*.

Here

☐ (i) implies that g does not create atoms in $\mathcal{E}^*$,

☐ (ii) implies that no compact set $\mathcal{A}^* \subset \mathcal{E}^*$ has infinite measure,

which are both needed for $\mathcal{P}^*$ to be Poisson.

Example 11 *If $\mathcal{P}$ is a homogeneous Poisson process of unit rate on $(0,\infty)$, and $g(x) = 1/x$, show that $g(\mathcal{P})$ is a Poisson process and find its intensity function. What if $g(x) = \lceil x \rceil$ or $g(x) = |\sin x|$?*

Note to Example 11

☐ The mean measure of $\mathcal{P}$ is given by $\mu[a,b] = (b-a)$, for $0 < a < b < \infty$.
(i) The function g maps $(0,\infty)$ to $(0,\infty)$, and $g \equiv g^{-1}$, so $g^{-1}(x^*) = 1/x^*$ satisfies $\mu[1/x^*, 1/x^*] = (1/x^* - 1/x^*) = 0$ for any $0 < x^* < \infty$.
(ii) Any compact set $\mathcal{A}$ of $(0,\infty)$ is a subset of a set $[a,b]$ for some $0 < a < b < \infty$, so

$$\mu(\mathcal{A}) = \mu(g^{-1}\mathcal{A}) = \int_{g^{-1}\mathcal{A}} \dot{\mu}(x)\,dx \le \int_{g^{-1}[a,b]} 1\,dx = \int_{1/b}^{1/a} dx = (1/a - 1/b) < \infty.$$

Hence $g(\mathcal{P})$ is indeed a Poisson process, and since $\mu[a,b] = (1/a - 1/b)$, its intensity function is $d\mu[a,b]/db = 1/b^2$, for $b > 0$.
A sketch shows what happens to the intensities of $\mathcal{P}$ and $g(\mathcal{P})$.

☐ With $g(x) = \lceil x \rceil$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x, then condition (i) fails whenever $x^* \in \mathbb{N}$, so the resulting process is not Poisson, as points of $\mathcal{P}^*$ could be superposed on the positive integers and thus $N^*(\cdot)$ is not well-defined. Equivalently,

$$\mu^*(\{n\}) = \mu[g^{-1}(\{n\})] = \mu\{(n-1, n]\} = 1, \quad n \in \mathbb{N},$$

so μ^* has atoms on every positive integer and thus is not diffuse.

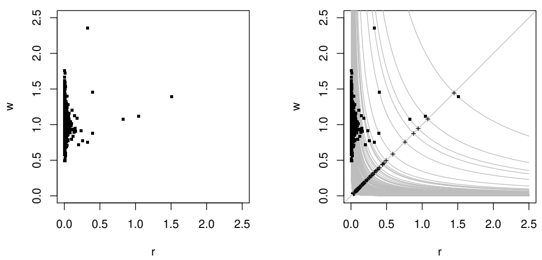
☐ With $g(x) = |\sin(x)|$ we have $\mathcal{E}^* = [0,1]$, and it is easy to check that while condition (i) is satisfied, $\mu^*([a,b]) = \infty$ for any $0 < a < b < 1$, so condition (ii) fails; $\mathcal{P}^*$ has an infinite number of points in any interval.

Example

Example 12 *Let $\mathcal{P}$ be a Poisson process with $\mathcal{E} = \mathbb{R}_+^2$ with $x = (r,w)$ generated by*

$$\mu\{(r,\infty) \times (w,\infty)\} = \frac{1}{r} \times \{1 - F(w)\}, \quad r, w > 0,$$

where $\sigma > 0$ and F is the CDF of a positive continuous random variable W with unit expectation. Show that $q = rw$ defines a Poisson process and find its intensity.



Left panel: first 1000 points (r,w) of a Poisson process sequentially generated on $\mathbb{R}_+^2$. Right panel: mapping of the points shown in the left panel to $q = rw$, shown as + on the diagonal, with the mapping function shown by the curved grey lines.

Note to Example 12

☐ In the picture $\mathcal{P} = \{(R_i, W_i) : i = 1, 2, \ldots\}$, where $R_1 > R_2 > \cdots > 0$ are generated sequentially by setting $R_i = (E_1 + \cdots + E_i)^{-1}$, with $E_i \overset{\text{iid}}{\sim} \exp(1)$, and $W_i = \exp(\sigma\varepsilon_i - \sigma^2/2)$, where $\varepsilon_i \overset{\text{iid}}{\sim} N(0, 1)$, independent of the E_i; note that $\mathrm{E}(W_i) = 1$. The first 1000 points of a realisation of such a process are shown in the left-hand panel of the figure; the full realisation would have an infinity of points at the left-hand edge of the panel, because $\mu\{(r, \infty) \times (0, \infty)\} = 1/r \to \infty$ as $r \to 0$.

☐ In the general case the mean measure has an *intensity function* $\dot{\mu}$ given by its derivative at the upper right corner of a rectangle $(r', r) \times (w', w)$, i.e.,

$$\begin{aligned} \dot{\mu}(r, w) &= \frac{\partial^2 \mu\{(r', r) \times (w', w)\}}{\partial r \partial w} \\ &= \frac{\partial^2}{\partial r \partial w} \{\mu(r', w') - \mu(r, w') - \mu(r', w) + \mu(r, w)\} \qquad (3) \\ &= \frac{1}{r^2} \times f(w), \quad r, w > 0, \qquad (4) \end{aligned}$$

where we have written $\mu(r, w) = \mu\{(r, \infty) \times (w, \infty)\}$ and so forth, and f denotes the density function corresponding to F.

☐ Let $g(r, w) = rw$, corresponding to setting $Q_i = R_i W_i$, which amounts to collapsing the points shown in the left-hand panel onto the diagonal line shown in the right-hand panel. For any $q > 0$, $\mu^*(q) = \mu[\{(r, q/r) : r > 0\}] = 0$ because μ has a density with respect to Lebesgue measure and the set $\{(r, q/r) : r > 0\}$ has Lebesgue measure zero, so this transformation does not create atoms. We can check the second property of μ^* once it is calculated. Note that $Q = RW > q$ if and only if $R > q/W$, and that $\mathcal{A}_q = \{(r, w) : rw > q\}$ has measure

$$\begin{aligned} \mu^*(q) = \mu(\mathcal{A}_q) &= \int_0^\infty f(w) \int_{r=q/w}^\infty \frac{1}{r^2} \, \mathrm{d}r \, \mathrm{d}w \\ &= \int_0^\infty f(w) \left[-\frac{1}{r}\right]_{q/w}^\infty \mathrm{d}w \\ &= \int_0^\infty f(w) \frac{1}{q/w} \, \mathrm{d}w \\ &= \frac{1}{q} \mathrm{E}(W) = \frac{1}{q}, \quad q > 0. \qquad (5) \end{aligned}$$

Hence $Q_i = R_i W_i$ is also Poisson, with the same mean measure as the R_i. This implies that the second property is also satisfied: any compact set $\mathcal{A}^*$ is a subset of (q_1, q_2) for some $q_2 > q_1$, so

$$\mu^*(\mathcal{A}^*) \le \mu^*(q_1, \infty) = \mu^*(q_2, \infty) = q_1^{-1} - q_2^{-1} < \infty.$$

☐ The restriction of $\mathcal{P}$ to a subset $\mathcal{E}'$ of $\mathcal{E}$ clearly also follows a Poisson process, with mean measure $\mu'(\mathcal{A}) = \mu(\mathcal{E}' \cap \mathcal{A})$. For example, if we let $\mathcal{E} = (0, \infty)$, consider $R_1, R_2, \ldots$ and let $\mathcal{E}' = (z', \infty)$ for some $z' > 0$, then we retain only those points R_i exceeding z'. As $\mu(\mathcal{E}') = 1/z'$ is finite, these R_i can be generated by first simulating a Poisson variable N' with mean $1/z'$, and if $N' = n$, simulating n independent variables on the interval (z', ∞) with survivor function z'/z; these Pareto variables have probability density function z'/z^2 $(z > z')$.

Marking

Theorem 13 (Marking) *Let $\mathcal{P}$ be a Poisson process on $\mathcal{E}$ with mean measure μ. Attach a random variable y_x, called the mark, to each point x of $\mathcal{P}$; the distribution of $y_x \in \mathcal{Y}$ may depend on x but not on any other point of $\mathcal{P}$. Then the points (x, y_x) form a Poisson process $\mathcal{P}^*$ in the product space $\mathcal{E} \times \mathcal{Y}$ with mean measure*

$$\mu(\mathcal{C}) = \iint_{(x,y)\in\mathcal{C}} \nu_x(\mathrm{d}y)\mu(\mathrm{d}x), \quad \mathcal{C} \subset \mathcal{E} \times \mathcal{Y},$$

where $\nu_x(\cdot)$ is the conditional probability measure of y_x given x.

- □ This provides an approach to making new Poisson processes, by attaching random variables to existing processes, and (perhaps) then applying the mapping theorem.
- □ If y_x takes a countable number of values ($\equiv$ colours), then the colouring theorem shows that the corresponding subsets of $\mathcal{P}$ are independent Poisson processes.

Note to Theorem 13

The Laplace functional of $\mathcal{P}^*$ is

$$\mathrm{E}\left\{\exp\left(-\int f \mathrm{d}\mathcal{P}^*\right)\right\} = \mathrm{E}_{\mathcal{P}}\left[\mathrm{E}\left\{\exp\left(-\int f \mathrm{d}\mathcal{P}^*\right)\right\} \,\middle|\, \mathcal{P}\right]$$

and the inner expectation on the right-hand side is

$$\prod_{x\in\mathcal{P}} \int_{\mathcal{Y}} e^{-f(x,y)}\, \nu_x(\mathrm{d}y) = \exp\left(-\int f^*\, \mathrm{d}\mathcal{P}\right),$$

say, where

$$f^*(x) = -\log \int_{\mathcal{Y}} e^{-f(x,y)}\, \nu_x(\mathrm{d}y).$$

Thus the Laplace functional of $\mathcal{P}^*$ is that of the Poisson process $\mathcal{P}$ with f replaced by f^*. But since

$$\begin{aligned}\int_{\mathcal{E}} \left\{1 - e^{-f^*(x)}\right\} \mu(\mathrm{d}x) &= \int_{\mathcal{E}} \left\{1 - \int_{\mathcal{Y}} e^{-f(x,y)}\, \nu_x(\mathrm{d}y)\right\} \mu(\mathrm{d}x) \\ &= \int_{\mathcal{E}} \int_{\mathcal{Y}} \left\{1 - e^{-f(x,y)}\right\}\, \nu_x(\mathrm{d}y)\mu(\mathrm{d}x),\end{aligned}$$

the result is established.

Basic result

Theorem 14 *Let* $X_1, \ldots, X_{nt_0} \stackrel{\text{iid}}{\sim} F$ *form* t_0 *blocks each of* n *observations, and suppose that sequences* $\{a_n\} > 0$ *and* $\{b_n\}$ *exist such that*

$$\mathrm{P}\left[\{\max(X_1, \ldots, X_n) - b_n\}/a_n \le x\right] \to G(x), \quad n \to \infty,$$

where G *is non-degenerate. Then as* $n \to \infty$ *the point processes*

$$\mathcal{P}_n = \{(j/(n+1), (X_j - b_n)/a_n) : j = 1, \ldots, nt_0\}$$

on $\mathcal{E} = [0, t_0] \times \mathcal{E}_x$ *converge in distribution to a Poisson process* $\mathcal{P}$ *with mean measure*

$$\mu\{(t', t) \times [x, \infty)\} = (t - t')\Lambda(x), \quad 0 \le t' < t \le t_0, \quad x \in \mathcal{E}_x = \{x' \in \mathbb{R} : \Lambda(x') < \infty\}, \tag{6}$$

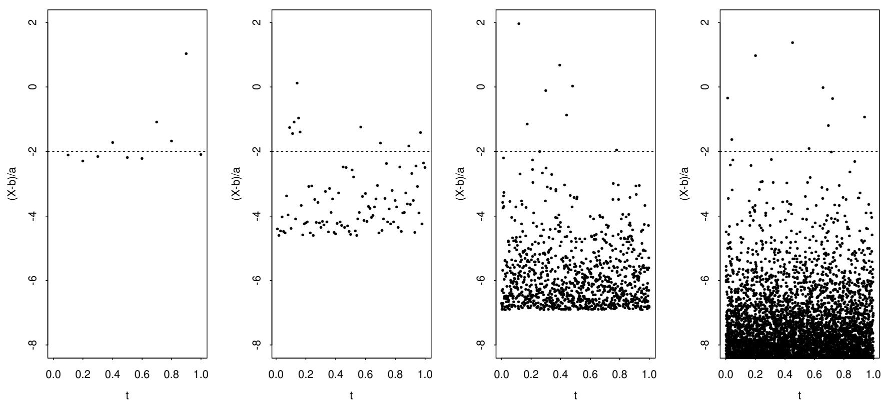
where

$$\Lambda(x) = \left(1 + \xi\frac{x - \eta}{\tau}\right)_+^{-1/\xi}$$

depends on parameters $\eta, \xi \in \mathbb{R}$ *and* $\tau > 0$ *and* $a_+ = \max(a, 0)$ *for real* a. *The corresponding intensity function is*

$$-\dot{\Lambda}(x) = \tau^{-1}\left(1 + \xi\frac{x - \eta}{\tau}\right)_+^{-1/\xi - 1} \ge 0.$$

Point process limit



☐ Here $\mathcal{E} \subset \mathbb{R}^D$, so we only need Kallenberg's theorem: for $\mathcal{A} \subset \mathcal{E}$, let $N_n(\mathcal{A}) = |\mathcal{P}_n \cap \mathcal{A}|$. Then if $\mathcal{B} \subset \mathcal{E}$ is any rectangle, and $\mathcal{C}$ is any finite union of disjoint rectangles of $\mathcal{E}$, and if

$$\mathrm{E}\{N_n(\mathcal{B})\} \to \mathrm{E}\{N(\mathcal{B})\}, \quad \mathrm{P}\{N_n(\mathcal{C}) = 0\} \to \mathrm{P}\{N(\mathcal{C}) = 0\}, \quad n \to \infty, \tag{7}$$

then $\mathcal{P}_n \stackrel{D}{\longrightarrow} \mathcal{P}$ as $n \to \infty$.

Forms of $\Lambda(x)$

- $\Lambda(x)$ is decreasing, but has three distinct forms:
  - when $\xi > 0$,

$$\Lambda(x) = \begin{cases} +\infty, & x \le \eta - \tau/\xi, \\ \left(1 + \xi \frac{x-\eta}{\tau}\right)_+^{-1/\xi}, & x > \eta - \tau/\xi, \end{cases}$$

    which is finite only for $x > \eta - \tau/\xi$, so $\Lambda(\mathcal{A}) = +\infty$, giving infinite counts, for any set $\mathcal{A}$ that goes below $\eta - \tau/\xi$;
  - for $\xi = 0$ we take the limit when $\xi \to 0$, giving

$$\Lambda(x) = \exp\{-(x-\eta)/\tau\}, \quad x \in \mathbb{R},$$

    which is finite for all x;
  - when $\xi < 0$,

$$\Lambda(x) = \begin{cases} \left(1 + \xi \frac{x-\eta}{\tau}\right)_+^{-1/\xi}, & x < \eta - \tau/\xi, \\ 0, & x \ge \eta - \tau/\xi, \end{cases}$$

    which is finite for all x.
- When $\xi \le 0$ the limiting mass at $-\infty$ is infinite, so any set $\mathcal{A}$ considered must have a finite lower bound.

Implications: Maxima

- A rescaled block maximum $Y_n = \{\max(X_1, \dots, X_n) - b_n\}/a_n$ satisfies

$$\begin{aligned} \mathrm{P}(Y_n \le y)] &= \mathrm{P}\left\{N_n\{(0,1) \times [y, \infty) = 0\right\} \\ &\to \mathrm{P}\left\{N\{(0,1) \times [y, \infty) = 0\right\} \quad n \to \infty, \\ &= \exp[-\mu\{(0,1) \times [y, \infty)\}], \\ &= \exp[\{-\Lambda(y)\}, \quad y \in \mathbb{R}, \end{aligned}$$

  so a block maximum has a limiting **generalized extreme-value (GEV)** distribution,

$$G(y) = \exp\left\{ -\left(1 + \xi \frac{y-\eta}{\tau}\right)_+^{-1/\xi} \right\}, \quad y \in \mathbb{R}. \tag{8}$$

- Let $Y_1 > \cdots > Y_r$ denote the r largest rescaled order statistics in a block, i.e., Y_1 is the maximum, etc. In the limit Y_2 is the largest of an infinite number of rescaled observations $(X_j - b_n)/a_n$, so its distribution is also G, but conditioned on $Y_2 < Y_1$. Hence

$$\mathrm{P}(Y_2 \le y_2 \mid Y_1 = y_1) = \exp\left\{\Lambda(y_1) - \Lambda(y_2)\right\}, \quad y_2 < y_1,$$

  and it follows that the limiting joint density of the **r-largest order statistics** $Y_r < \cdots < Y_1$ is

$$f(y_1, \dots, y_r) = \exp\{-\Lambda(y_r)\} \prod_{j=1}^{r} \{-\dot{\Lambda}(y_j)\}, \quad y_r < \cdots < y_1. \tag{9}$$

Implications: Threshold exceedances

- ☐ Consider the 'forgetting' mapping $g : \mathbb{R}^2 \to \mathbb{R}$ with $g(t,x) = x$, giving the process of event sizes $\mathcal{P}^* = g(P)$ without their times. The mapping theorem (Theorem 10) implies that $\mathcal{P}^*$ is Poisson with mean measure

$$\mu^*\{[x,\infty)\} = \mu[g^{-1}\{[x,\infty)\}] = \mu\{[0,t_0] \times [x,\infty)\} = t_0\{1+\xi(x-\eta)/\tau\}_+^{-1/\xi}.$$

- ☐ For $y \in \mathbb{R}$, let $\mathcal{A}_y = [y,\infty)$ and let $N^*(\mathcal{A}_y) = |\mathcal{P}^* \cap \mathcal{A}_y|$.
- ☐ The conditional property (Theorem 6) implies that conditional on $N^*(\mathcal{A}_u) = n$, these n threshold exceedances have the same distribution as n points generated independently on $\mathcal{A}_u$ with measure

$$\frac{\mu^*(\mathcal{A}_{u+x})}{\mu^*(\mathcal{A}_u)} = \frac{t_0\{1+\xi(x-u-\eta)/\tau\}_+^{-1/\xi}}{t_0\{1+\xi(u-\eta)/\tau\}_+^{-1/\xi}} = \left(1+\xi\frac{x}{\sigma_u}\right)_+^{-1/\xi}, \quad x > 0,$$

where $\sigma_u = \tau + \xi(u-\eta)$.

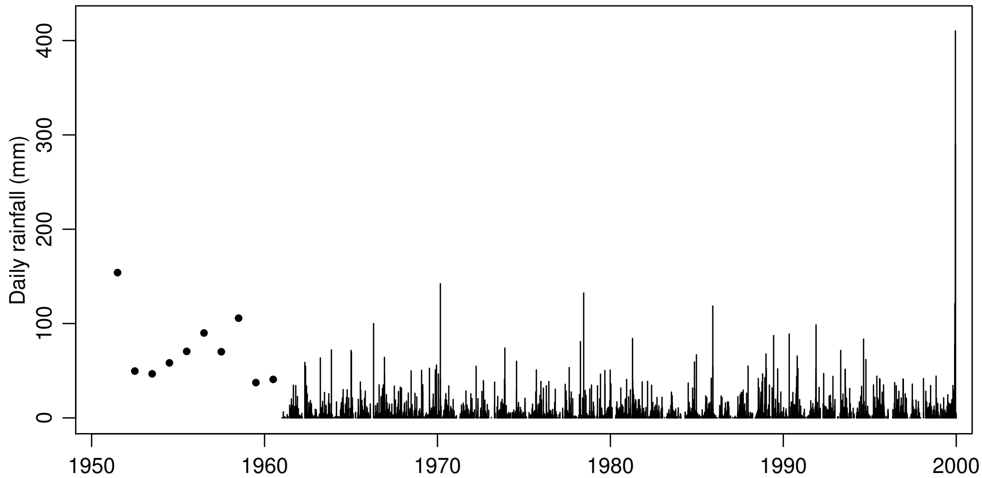
- ☐ Hence, provided that $\sigma_u > 0$, exceedances of the threshold u arise with rate $\Lambda(u)$ and are independent with the **generalized Pareto distribution (GPD)**

$$H(x) = 1 - \left(1+\xi\frac{x}{\sigma_u}\right)_+^{-1/\xi}, \quad x > 0, \tag{10}$$

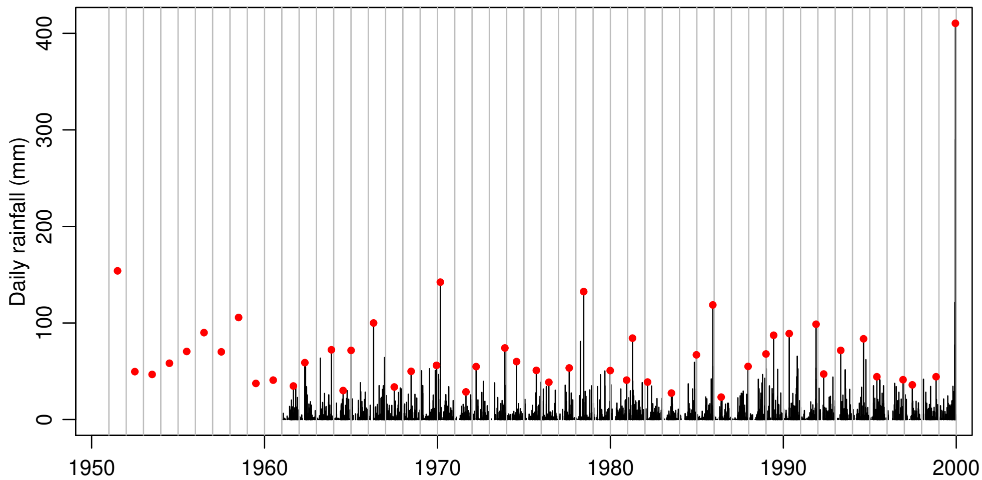
Statistical applications

- ☐ Theorem 14 gives the basic models for univariate extremes:
  - (quite often) we analyse block maxima by fitting the GEV (4);
  - (less often) we analyse the r largest observations in a block by fitting model (5);
  - (quite often) we analyse threshold exceedances either
    - ▷ by fitting the basic Poisson process model with mean measure (2) or
    - ▷ by fitting threshold exceedances using the GPD (6).
  - We can 'mix and match' these models if necessary — in the next slide there are annual maxima for the first 10 years, then daily values, so a likelihood can be constructed using the GEV for the maxima and then the GPD for exceedances of a suitable threshold.
- ☐ In all cases statistical questions arise:
  - what should we try and estimate, and how?
  - how do we known whether our assumptions are OK?
  - the asymptotic models will be fitted to finite-sample data — does this introduce bias?
  - are our conclusions robust to model failure?
- ☐ In the next chapters we attempt to answer these questions.
- ☐ The Poisson process viewpoint will reappear later.

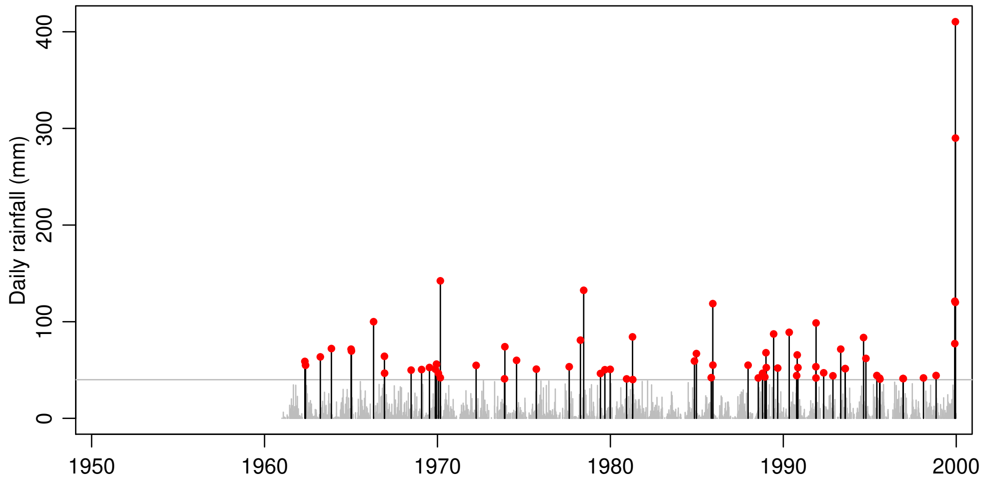
Vargas data



Vargas maxima



Vargas exceedances



Note 1 to Theorem 14

☐ If a limiting distribution G for rescaled maxima exists, then

$$\begin{aligned} \mathrm{P}[\{\max(X_1,\ldots,X_n)-b_n\}/a_n \le y] &= \mathrm{P}\{\max(X_1,\ldots,X_n) \le b_n + a_n y\} \\ &= F^n(b_n+a_n y) \\ &= \left[1 - \frac{n\{1-F(b_n+a_n y)\}}{n}\right]^n. \end{aligned}$$

Hence a limiting function $\Lambda(y)$ must exist such that

$$\Lambda_n(y) = n\{1-F(b_n+a_n y)\} \to \Lambda(y), \quad n \to \infty.$$

☐ Let $H(x) = -\log\{1-F(x)\}$ denote the *cumulative hazard function* corresponding to F, and choose $b_n = b_n^*$ such that $H(b_n^*) = -\log n$, so that

$$\log \Lambda_n(y) = H(b_n + a_n y) - H(b_n).$$

☐ We suppose that F is continuous, places probability in an interval $[x_*, x^*]$, where either or both limits might be infinite, F is not defective (so there is no mass at x^*), that H is twice continuously differentiable with reciprocal hazard function $r(x) = 1/H'(x)$, and that $\lim_{x\to x^*} r'(x) = \xi$ is real and finite. These are sometimes called the *von Mises conditions*.

☐ Then

$$H(b_n+a_n y) - H(b_n) = a_n \int_0^y \frac{1}{r(b_n+a_n x)}\,\mathrm{d}x = a_n \int_0^y \frac{1}{r(b_n) + a_n x r'\{b_n + s_n(x)\}}\,\mathrm{d}x,$$

where $s_n(x)$ lies between zero and x. If we now choose $a_n = a_n^* = r(b_n^*)$, which is positive because $r(x) = \{1-F(x)\}/f(x)$, we have

$$H(b_n^* + a_n^* y) - H(b_n^*) = \int_0^y \frac{1}{1 + x r'\{b_n^* + s_n(x)\}}\,\mathrm{d}x = \int_0^y \frac{1}{1+\xi_n x} g_n(x)\,\mathrm{d}x,$$

where $\xi_n = r'(b_n^*)$ and $g_n(x) = (1+\xi_n x)/\{1 + x r'\{b_n^* + s_n(x)\}$.

☐ The implicit function theorem implies that $s_n(x)$ is continuous in x and so is r', so $g_n(x)$ is continuous in x, and one can check that $g_n(x) \to 1$ as $n \to \infty$. Hence in the interval where $1 + \xi_n x$ does not change sign, we can use a mean value theorem for integrals and choose y^* such that

$$H(b_n^* + a_n^* y) - H(b_n^*) = g_n(y^*) \int_0^y \frac{1}{1+\xi_n x}\,\mathrm{d}x = g_n(y^*) \times \xi_n^{-1} \log(1+\xi_n x)_+,$$

where we add the $(\cdot)_+$ to remind us that the term in brackets must be positive. Now $\xi_n = r'(b_n^*) \to \xi$ and $0 < y^* < y$ as $n \to \infty$, so

$$\lim_{n\to\infty} H(b_n^* + a_n^* y) - H(b_n^*) = \xi^{-1} \log(1+\xi y)_+ = \log \Lambda(y),$$

as required. This establishes sufficient conditions under which a maximum has limiting distribution $\exp\{-\Lambda(y)\}$, with $\eta = 0$ and $\tau = 1$. We need the more general case to allow for the fact that b_n and a_n are unknown in applications (because F is unknown).

Note 2 to Theorem 14

☐ To establish the Poisson convergence, define the binomial processes

$$\mathcal{P}_n = \{(j/(n+1), (X_j - b_n)/a_n) : j = 1, \ldots, nt_0\}, \quad n = 1, 2, \ldots,$$

and the corresponding count process $N_n(\cdot)$ on $\mathcal{E} = [0, t_0] \times \mathcal{E}_x$.

☐ Let $0 < t_1 < t_2 \leq t_0$ and $x_1 < x_2$ determine the rectangle $\mathcal{A} = (t_1, t_2] \times (x_1, x_2]$, let

$$\mu(\mathcal{A}) = (t_2 - t_1)\{\Lambda(x_1) - \Lambda(x_2)\}, \quad \mathcal{A} \subset \mathcal{E},$$

and let $\mathcal{P}$ denote a Poisson process on $\mathcal{E}$ with mean measure μ.

☐ We now check Kallenberg's conditions. If $\lfloor x \rfloor$ is the integer part of x, then

$$\begin{aligned} \mathrm{E}\{N_n(\mathcal{A})\} &= \lfloor (n+1)t_2 - (n+1)t_1 \rfloor \times \mathrm{P}\{x_1 < (X_j - b_n)/a_n \leq x_2\} \\ &= \frac{\lfloor (n+1)(t_2 - t_1) \rfloor}{n} \times \Lambda_n(x_1, x_2] \\ &\to (t_2 - t_1)\Lambda(x_1, x_2] = \mu(\mathcal{A}), \quad n \to \infty, \end{aligned}$$

which verifies the first condition.

☐ For the second condition, let $\mathcal{C}$ be a union of a finite number of disjoint rectangles of $\mathcal{E}$, and note that we can write $\mathcal{C} = \bigcup_{i=1}^k \mathcal{T}_i \times \bigcup_{l=1}^{L_i} \mathcal{X}_{i,l}$, where the $\mathcal{T}_i \subset [0, t_0]$ are disjoint intervals, and the intervals $\mathcal{X}_{i,l} \subset \mathbb{R}$ are disjoint for each i. Let $\mathcal{T}_1 = (t_1, t_2]$, let $\mathcal{X}_i = \bigcup_{l=1}^{L_i} \mathcal{X}_{i,l}$ and $\mathcal{B}_i = \mathcal{T}_i \times \mathcal{X}_i$, and note that independence and identical distribution of the X_j gives

$$\begin{aligned} \mathrm{P}\{N_n(\mathcal{B}_1) = 0\} &= \mathrm{P}\{(X_1 - b_n)/a_n \notin \mathcal{X}_1\}^{\lfloor (n+1)(t_2 - t_1) \rfloor} \\ &= \left[\left\{1 - \frac{\Lambda_n(\mathcal{X}_1)}{n}\right\}^n\right]^{\lfloor (n+1)(t_2 - t_1) \rfloor / n} \\ &\to \exp\{-|\mathcal{T}_1|\Lambda(\mathcal{X}_1)\}, \quad n \to \infty, \\ &= \exp\{-\mu(\mathcal{T}_1 \times \mathcal{X}_1)\}. \end{aligned}$$

This applies for each $\mathcal{T}_i$, and the corresponding variables X_j are independent, so

$$\begin{aligned} \mathrm{P}\{N_n(\mathcal{C}) = 0\} &= \prod_{i=1}^k \mathrm{P}\{N_n(\mathcal{B}_i) = 0\} \\ &\to \prod_{i=1}^k \exp\{-\mu(\mathcal{T}_i \times \mathcal{X}_i)\}, \quad n \to \infty, \\ &= \exp\left\{-\sum_{i=1}^k \mu(\mathcal{B}_i)\right\} \\ &= \exp\{-\mu(\mathcal{C})\}, \end{aligned}$$

which establishes the second condition. Thus $\mathcal{P}_n \stackrel{D}{\longrightarrow} \mathcal{P}$.