Final exam – Solutions

Graph Theory 2019 – EPFL – István Tomon

1. *Prove that in any connected graph G, there is a walk that contains every edge of G exactly twice.*

   Solution. We duplicate each edge of G in order to get the new multigraph G'. As G was connected, G' is also connected. Also, all vertices of G' have even degree by construction, so G' has an Euler tour. This tour corresponds to a walk on G that uses every edge exactly twice, as needed.

2. *Prove Dirac's theorem: If a graph G on $n \geq 3$ vertices has minimum degree at least $\frac{n}{2}$, then it contains a Hamilton cycle.*

   Solution. First observe that G must be connected, because each component contains at least $\delta(G) + 1 > n/2$ vertices, so G cannot have more than one components.

   Take a longest path $P = v_1 v_2 \dots v_k$ in G. By maximality, all neighbors of v_1 and v_k are in the path. Let us say that an edge $v_i v_{i+1}$ is type-1 if $v_{i+1} \in N(v_1)$, and let us say that it is type-2 if $v_i \in N(v_k)$. As $\delta(G) \geq n/2$, we have at least $n/2$ type-1 and $n/2$ type-2 edges in P. But P has at most $n-1$ edges, so some edge $v_j v_{j+1}$ is both type-1 and type-2, i.e., $v_1 v_{j+1}$ and $v_j v_k$ are edges of G. Then $C = P - v_j v_{j+1} + v_1 v_{j+1} + v_j v_k = v_j \dots v_1 v_{j+1} \dots v_k v_i$ is a cycle.

   In fact, C is a Hamilton cycle. Indeed, suppose not all vertices are contained in C. Since G is connected, there must be an edge uv_i where $u \notin C$. Then there is a path that goes from u to v_i and then all around the cycle C to a neighbor of v_i. This path contains $k+1$ vertices, contradicting the maximality of P.

3. *Prove that every connected planar graph on $n \geq 3$ vertices has a triangular face or a vertex of degree at most 3. (A triangular face is one whose boundary has length 3).*

   Solution. If there is a triangular face in the graph, we are done. So suppose every face of the graph is not triangular. As $n \geq 3$, this means that the length ℓ_F of each face F is at least four. Then the sum of ℓ_F (over all the faces F) is at least $4f$ (where f is the number of faces). On the other hand, this sum is $2e$ (where e is the number of edges) because each edge is counted twice by the sum. Therefore $4f \leq 2e$, i.e., $f \leq e/2$. Combining this with Euler's formula, this gives $2 = n - e + f \leq n - e/2$, so $e \leq 2n - 4$, which implies that the sum of degrees of all the vertices is at most $4n - 8$. Thus there must be a vertex of degree at most 3, as required.

4. *Let G be a bipartite graph with parts of size $2n$ and minimum degree at least n. Prove that G has a perfect matching.*

   Solution. Let G have parts A and B. We will check Hall's condition for A. Take $X \subseteq A$. If X is empty, then $|N(X)| = |X| = 0$, so the condition is satisfied. If $1 \leq |X| \leq n$, then $|N(X)| \geq n \geq |X|$, because any vertex in X has at least n neighbors in B. Finally, if $|X| > n$, then $N(X) = B$ because every vertex v in B has at least n neighbors in A, so it must have a neighbor in X, as well. (Otherwise $X \cup N(v)$ would contain more than $2n$ distict vertices in A). In particular, $|N(X)| = 2n \geq |X|$, so the condition holds for every X. By Hall's theorem, there is a perfect matching.

5. *Prove that if a graph G on n vertices does not contain $K_{2,2}$ as a subgraph, then G has at most $n^{3/2}$ edges.*

   Solution. Let G be a graph on n vertices without a 4-cycle. Let S be the set of "cherries", i.e., pairs $(u, \{v, w\})$ where u is adjacent to both v and w, with $v \neq w$:

   We will count the elements of S of in two different ways. Summing over u, we find $|S| = \sum_{u \in V(G)} \binom{d(u)}{2}$. On the other hand (and this is the crucial observation): every pair $\{v, w\}$ has at most one common neighbor (because G is $K_{2,2}$-free), so $|S| \leq \binom{n}{2}$.

   The rest of the proof is just calculations. So far we have $\sum_{u \in V} \binom{d(u)}{2} \leq \binom{n}{2}$ or equivalently,

   $$\sum_{u \in V} d(u)^2 \leq n(n-1) + \sum_{u \in V} d(u).$$

   Using Cauchy-Schwarz or AM-QM, we have $\left(\sum_{u \in V} d(u)\right)^2 \leq n \sum_{u \in V} d(u)^2$. This, together with (**??**), implies

   $$\left(\sum_{u \in V} d(u)\right)^2 \leq n^2(n-1) + n \sum_{u \in V} d(u).$$

   Here the sum of the degrees is $2|E(G)|$, so we get $4|E(G)|^2 \leq n^2(n-1) + 2n|E(G)|$. Or equivalently,

   $$|E(G)|^2 - \frac{n}{2}|E(G)| - \frac{n^2(n-1)}{4} \leq 0.$$

   The left-hand side is a quadratic function of $|E(G)|$ which is increasing whenever $|E(G)| > n/2$ and positive for $|E(G)| = n^{3/2}$, so the inequality can only be true if $|E(G)| < n^{3/2}$.

6. *Let $n \geq 2$ be an integer, and $R(n, n)$ be the corresponding Ramsey number. Show that any sequence of $N \geq R(n, n)$ distinct numbers $a_1, \ldots, a_N$ contains a monotone (increasing or decreasing) subsequence of length n.*

   Solution. Let us color the edges of the complete graph on $[N]$ as follows. Color the edge ij (with $i < j$) blue if $a_i < a_j$ and red otherwise. By the definition of $R(n, n)$, and the fact that $N \geq R(n, n)$, we know that this graph contains a monochromatic clique of size n, say induced by the vertices $i_1, \ldots, i_n$ (where $i_1 < \cdots < i_n$). In particular, the edges $i_j i_{j+1}$ have the same color for all $j = 1, \ldots, n-1$. If the color is blue, then this means that $a_{i_1}, \ldots, a_{i_n}$ form an increasing subsequence, if the color is red then $a_{i_1}, \ldots, a_{i_n}$ form a decreasing subsequence. Either way, we have a monotone subsequence of length n.

7. *Let G be a bipartite graph. Prove that if λ is an eigenvalue of the adjacency matrix of G, then $-\lambda$ is also an eigenvalue.*

   Solution. As G is bipartite, for some $s \times t$ matrix B we have

   $$A_G = \begin{bmatrix} O_{s \times s} & B \\ B^T & O_{t \times t} \end{bmatrix}.$$

   Since λ is an eigenvalue, we have

   $$\begin{bmatrix} \lambda v \\ \lambda w \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix} = A_G \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} Bw \\ B^T v \end{bmatrix}$$

for some vector $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$. So $Bw = \lambda v$ and $B^T v = \lambda w$. But then

$$A_G \begin{bmatrix} v \\ -w \end{bmatrix} = \begin{bmatrix} -Bw \\ B^T v \end{bmatrix} = -\lambda \begin{bmatrix} v \\ -w \end{bmatrix}.$$

Thus $-\lambda$ is also an eigenvalue.

8. *Let G be a graph, and suppose $d \geq 0$ is the smallest number such that G is d-degenerate. Prove that G has at least $\frac{d(d+1)}{2}$ edges.*
(A graph is d-degenerate if each of its subgraphs has a vertex of degree at most d.)

 Solution. We know that G is not $d-1$-degenerate, so it contains a subgraph H of minimum degree at least d. Any graph of minimum degree d has at least $d+1$ vertices, so $|V(H)| \geq d+1$. This means that the sum of the degrees in H is at least $d|V(H)| \geq d(d+1)$, and therefore H has at least $\frac{d(d+1)}{2}$ edges and, of course, so does G.

9. *Prove the fan lemma: Let k be a positive integer. If G is a k-connected graph, then for every vertex s, and for every set T of at least k vertices, there are k paths from s to T in G that are vertex-disjoint except for their starting vertex s.*

 Solution. Add a vertex t that is adjacent to all the vertices in T and call the resulting graph G'. We first check that G' is k-connected, i.e., that deleting a set X of at most $k-1$ vertices keeps it connected.

 Indeed, if $t \in X$, then $G' - X = G - Y$ for some Y of size at most $k-2$, which is connected by the k-connectivity of G. Otherwise, $G' - X$ can be obtained from $G - X$ (which is again connected for the same reason), by adding the new vertex t to it and connecting it to the remaining neighbors. As t has at least k neighbors, not all of them are deleted, so $G' - X$ is still connected.

 Now we can apply Menger's theorem to G' and obtain k internally vertex-disjoint paths from s to t. Removing t from these paths gives paths from s to T that are disjoint aside from s.

10. *Let n and k be positive integers. Show that the edges of K_n can be colored with k colors so that the number of monochromatic triangles is at most $\frac{1}{k^2}\binom{n}{3}$.*
(A monochromatic triangle is a 3-cycle whose edges have the same color.)

 Solution. We will show that a random coloring works. So let X be a random variable counting the number of monochromatic triangles in a random coloring of the edges of K_n with k colors, and let X_T be a random variable taking value 1 if a given triangle T is monochromatic, and 0 otherwise. Since the total number of possible colorings of T is k^3, and there are k ways to color T as a monochromatic triangle, we have that $\mathbb{E}[X_T] = k/k^3$.

 Since $X = \sum_T X_T$, by the linearity of expectation (and since there are $\binom{n}{3}$ possible distinct triangles in K_n), we have that

$$\mathbb{E}[X] = \mathbb{E}[\sum_T X_T] = \sum_T \mathbb{E}[X_T] = \binom{n}{3} \cdot \frac{1}{k^2}.$$

 Thus, there exists a coloring of K_n where the number of monochromatic triangles is at most $\frac{\binom{n}{3}}{k^2}$.