Final exam – Solutions

Graph Theory 2018 – EPFL – Dániel Korándi

1. *Prove that if G is a connected planar graph on n vertices that has finite girth g, then it has at most $\frac{g}{g-2}(n-2)$ edges.*

   Solution. Fix a planar drawing D of G. By double counting the pairs (e, f) where e is an edge on the boundary of face f, we get $2|E(G)| \geq g|F_D(G)|$. Euler's formula says $|V(G)|-|E(G)|+|F_D(G)| = 2$ (G is connected!). Plugging in the bound on $|F_D(G)|$ gives

$$n - |E(G)| + \frac{2}{g} \cdot |E(G)| \geq 2 \qquad \Rightarrow \qquad \frac{g}{g-2}(n-2) \geq |E(G)|.$$

2. *Show that in any tree containing an even number of edges, there is at least one vertex with even degree.*

   Solution. Suppose not, i.e., all vertices have odd degree. The sum of the degrees is always even, so we must have an even number of vertices. But the number of edges in a tree is 1 fewer than the number of vertices, so such a tree has an odd number of edges.

3. *Prove that a K_3-free graph on n vertices contains at most $\lfloor \frac{n^2}{4} \rfloor$ edges.*

   Solution. Let v be a vertex in the graph G of maximum degree Δ, and let $S = N(v)$ be its neighborhood (so $|S| = \Delta$). Note that there is no edge with both endpoints in S, otherwise we would get a triangle with v. So every edge of G touches a vertex in $V(G) \setminus S$. Also, every vertex touches at most Δ edges, so the total number of edges is at most

$$\Delta|V(G) \setminus S| = \Delta(n - \Delta) \leq \lfloor n^2/4 \rfloor.$$

   (At the inequality in the middle, we used the easy fact that $ab \leq (\frac{a+b}{2})^2$.)

4. *Let G be a connected graph with maximum degree Δ, such that $\chi(G) = \Delta + 1$. Prove that G is Δ-regular.*

   Solution. We have seen in the lectures that if G is d-degenerate, then $\chi(G) \leq d + 1$. So our G is not $(\Delta - 1)$-degenerate. Then, by definition, G has a subgraph H with minimum degree $\delta(H) > \Delta - 1$. But we know $\Delta(H) \leq \Delta(G) = \Delta$, so H is Δ-regular.

   Now the only way a connected graph G with maximum degree Δ can contain a Δ-regular subgraph H is if $H = G$. Indeed, if $H \subsetneq G$, then there is an edge in G that touches a vertex v of H, but is not in H. But then $d_G(v) > d_H(v) = \Delta$, a contradiction.

5. (a) *[**7 points**] Show that if for some real number $0 \leq p \leq 1$ we have $\binom{n}{s}p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$, then $R(s, t) > n$.*

   (b) *[**3 points**] Deduce that there is a positive constant c such that $R(4, t) \geq c \cdot \frac{t^{3/2}}{\log^{3/2} t}$ for every integer $t \geq 2$.*

   Solution.

   (a) Consider the random coloring of the edges of K_n by red and blue, such that each edge is colored independently by red with probability p, and by blue with probability $1-p$. Then the expected number of red s-cliques is $\binom{n}{s}p^{\binom{s}{2}}$, and the expected number of blue t-cliques is $\binom{n}{t}(1-p)^{\binom{t}{2}}$. Therefore, the total expectation of red K_s's and blue K_t's is $m := \binom{n}{s}p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}}$. Here $m < 1$ by assumption, so there is a coloring without any red K_s or blue K_t. Therefore, we have $R(s, t) > n$.

(b) We want to make $\binom{n}{4}p^{\binom{4}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}}$ less than 1 for large n. $\binom{n}{4} < n^4/2$, so for $p = n^{-2/3}$ we have $\binom{n}{4}p^{\binom{4}{2}} < n^4p^6/2 = 1/2$. For this p the second term is

$$\binom{n}{t}(1-p)^{\binom{t}{2}} \leq n^t e^{-p\binom{t}{2}} \leq \exp(t\log n - n^{-2/3}t^2/4)$$

If $n \leq c\frac{t^{3/2}}{\log^{3/2} t}$ for some small enough c then $n^{-2/3}t^2/4 > t\log n + 1$, hence this term is also less than $1/2$ and we can apply part (a).

6. *Prove that a connected graph has an Eulerian tour if and only if each vertex has even degree.*

Solution. The proof is based on the following claim.

Claim. *In a graph where all vertices have even degree, every maximal trail is a closed trail.*

Proof. Let T be a maximal trail. If T is not closed, then T has an odd number of edges incident to the final vertex v. However, as v has even degree, there is an edge touching v that is not contained in T. This edge can be used to extend T to a longer trail, contradicting the maximality of T. □

Now we are ready to prove the statement. To see that the condition is necessary, suppose G has an Eulerian tour C. If a vertex v was visited k times in the tour C, then each visit used 2 edges incident to v (one incoming edge and one outgoing edge). Thus, $d(v) = 2k$, which is even.

To see that the condition is sufficient, let G be a connected graph with even degrees. Let $T = e_1e_2\dots e_\ell$ (where $e_i = (v_{i-1}, v_i)$) be a longest trail in G. Then it is maximal, of course. According to the Lemma, T is closed, i.e., $v_0 = v_\ell$. G is connected, so if T does not include all the edges of G then there is an edge e outside of T that touches it, i.e., $e = uv_i$ for some vertex v_i in T. Since T is closed, we can start walking through it at any vertex. But if we start at v_i then we can append the edge e at the end: $T' = e_{i+1}\dots e_\ell e_1 e_2 \dots e_i e$ is a trail in G which is longer than T, contradicting the fact that T is a longest trail in G. Thus, T must include all the edges of G and so it is an Eulerian tour.

7. *Let A be an $n \times m$ matrix of non-negative real numbers such that the sum of the entries is an integer in every row and in every column. Prove that there is an $n \times m$ matrix B of non-negative integers such that in every row and in every column, the sum of the entries in B is the same as in A.*

Solution. Let $a_1, \dots, a_n$ and $b_1, \dots, b_m$ be the sums of the entries in the rows and columns of A, respectively. Consider the complete bipartite graph $G = (P \cup Q, E)$, where $P = \{p_1, \dots, p_n\}$ and $Q = \{q_1, \dots, q_m\}$. Let us make a directed network out of G. Orient all edges from P to Q, and set their capacity to infinity. Add a source s, and an oriented edge (s, p_i) of capacity a_i for every $p_i \in P$. Analogously, add a sink t, and an oriented edge (q_j, t) of capacity b_j for every $q_j \in Q$.

It is easy to check that the $(s, V-s)$ and $(V-t, t)$ are the only minimum cuts, and they have capacity $a_1 + \dots + a_n = b_1 + \dots + b_m$. Since all capacities are integral, the Ford-Furkerson theorem shows that there is an integer flow with this value. Then the matrix B where each entry B_{ij} is defined to be the value of this flow on the edge p_iq_j will have the same row and column sums as A.

8. *Describe an efficient algorithm for finding a minimum-weight spanning tree in a connected weighted undirected graph, and prove that it indeed returns such a tree.*

Solution. One possibility is the Tree-Growing (or Prim's) Algorithm

Tree-Growing Algorithm for connected weighted undirected graphs

(a) Start with T being a single vertex;

(b) Repeat the following until $V(T) = V(G)$:

  i. Find $e \in \partial(T)$ with minimum $w(e)$ (here $\partial(T) =$ edges leaving T);

  ii. Add e to T.

(c) Return T.

Let us prove that it returns a minimum spanning tree. Call a tree *good* if it is contained in a minimum spanning tree. Clearly the empty graph is good. We will inductively show that all the trees occurring during the algorithm are good, hence so is the final tree T. Since the final tree T has $V(T) = V(G)$, it is itself a spanning tree, so it follows that it has minimum weight.

Suppose we are in the second step of the algorithm with a good tree T', contained in a minimum spanning tree T^*. Then we want to show that $T' + e$ is also good, where $e \in \partial(T')$ with minimum $w(e)$. If $e \in T^*$, then this is obvious, so suppose $e \notin T^*$. Adding e to T^* creates a cycle, which must contain another edge $f \in \partial(T')$. Since the algorithm chose e over f, we have $w(f) \geq w(e)$. Then $T^{**} = T^* - f + e$ is a minimum spanning tree containing T' and e, which means $T' + e$ is good.

9. *Let G be a k-connected graph with at least $2k$ vertices for some $k \geq 2$.*

   *(a) [**5 points**] Prove that G contains a cycle of length at least k.*

   *(b) [**5 points**] Prove that G contains a cycle of length at least $2k$.*

Solution.

(a) We have seen in the lectures that $\delta(G) \geq \kappa(G)$, and we had a problem on problem set 1 that showed that every graph contains a cycle of length at least $\delta(G)+1$. Our assumption of $\kappa(G) \geq k$ then gives a cycle of length at least $k+1$, even.

(b) Take a longest cycle C in G. We already know it has length at least k, but suppose its length is less than $2k$. Then there is a vertex v not in C. By the fan lemma from problem set 9 (and the k-connectivity of G), there are k vertex disjoint v-C paths in G. Since C contains at most $2k-1$ vertices, two of these paths (say, P and P') connect v to consecutive vertices of C (say, v and v'). But then we can replace the edge vv' in C by the path $P \cup P'$ to get a longer cycle, contradicting our assumption.