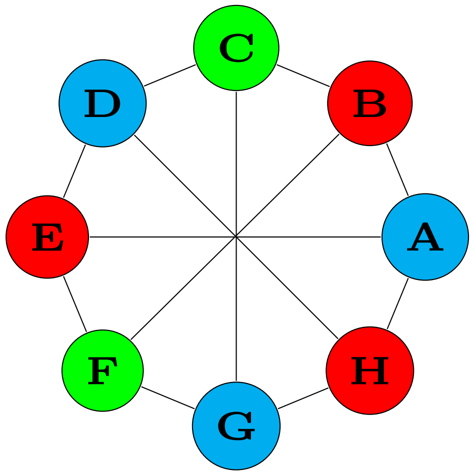
Graph Theory
MATH - 360

EPFL - Fall Semester 2024/2025

Solutions for the exam

Question 1: Let G be the Wagner graph:

- diameter: $\mathrm{diam}(G) = 2$ (any x, y are connected by an edge or a 2-long path);
- girth: $\mathrm{g}(G) = 4$ (A-B-F-E is a smallest cycle);
- independence number: $\alpha(G) = 3$ (see figure);
- chromatic number: $\chi(G) = 3$ (see figure);
- G is not planar (contains a subdivision of K_3);
- G is not Eulerian (all vertices have odd degrees);
- G is Hamiltonian (A-B-C-D-H-G-F-E-A is a Hamilton cycle)



A proper 3-coloring of G

Question 2: G is a connected graph, therefore it has a spanning tree T (Theorem 3.1). By definition of a spanning tree, T has the same number of vertices as G, that is $n \geq 2$. Hence T has two leaves (Lemma 2.7). Consider $v \in V(T)$ one such leaf. By definition of a leaf, $T - v$ is connected. Furthermore $T - v$ is a subgraph of $G - v$. Thus $G - v$ is connected. The leaf $v \in V(T) = V(G)$ is the vertex of G that we are looking for.

Question 3: Consider a longest such path $\mathcal{P} = v_k \dots v_1 w u_1 \dots u_l$, where $v_k \dots v_1 w$ is a red path and $w u_1 \dots u_l$ is a blue path. We reason by contradiction and assume that $\mathcal{P}$ is not Hamiltonian. Then there is a vertex x not contained in it. Consider the edge wx. If it is red, then the path $v_k \dots v_1 w x u_1 \dots u_l$ satisfies the required property, and it is longer than $\mathcal{P}$ (no matter if xu_1 is red or blue). Similarly, if wx is blue, then $v_k \dots v_1 x w u_1 \dots u_l$ is a longer such path. Hence if $\mathcal{P}$ is not Hamiltonian, we can build a longer path that is the union of two monochromatic ones, which is a contradiction.

Alternatively, one can prove this statement by induction on n, using the same idea of looking at wx.

Question 4: Take a longest path $v_0 v_1 \dots v_\ell$ in G, which is of length ℓ. Suppose that $\ell < k$. If v_0 and v_ℓ are not adjacent, then by assumption $d(v_0) + d(v_\ell) \geq k$. By maximality, all neighbors of v_0 and v_ℓ are in the path. Let us now define two types of edges: for $i \in \{1, \dots, \ell - 1\}$, an edge $v_i v_{i+1}$ is of type 1 if $v_{i+1} \in N(v_0)$ and is of type 2 if $v_i \in N(v_\ell)$. Since $d(v_0) + d(v_\ell) \geq k > \ell$, there exists an edge $v_i v_{i+1}$ which is both of types 1 and 2. Hence we get a cycle $v_i \dots v_0 v_{i+1} \dots v_\ell$ of length $\ell + 1$. In the case where v_0 and v_ℓ are adjacent, $v_0 v_1 \dots v_\ell v_0$ is a cycle of length $\ell + 1$. Now, in both cases, since the number of vertices in the cycle is $\ell + 1 < n$, there exists a vertex u not in the cycle. By connectedness of G, there is an edge uv_j where v_j, $j \in \{0, \dots, \ell\}$ is in the longest path. Then we can longer path by adding u, which leads to a contradiction.

Question 5: Let $n < 12$ be the number of vertices of G. Since G is planar, the corollary of Euler's formula (Proposition 5.3) gives: $|E(G)| \leq 3(n-6)$. Now, by the handshakes formula, we have: $\sum_{v \in V(G)} \deg(v) = 2|E(G)| \leq 2(3n-6) = 6n - 12 < 6n - n = 5n$. Finally, by the pigeonhole principle, this further implies that G has a vertex of degree at most 4.

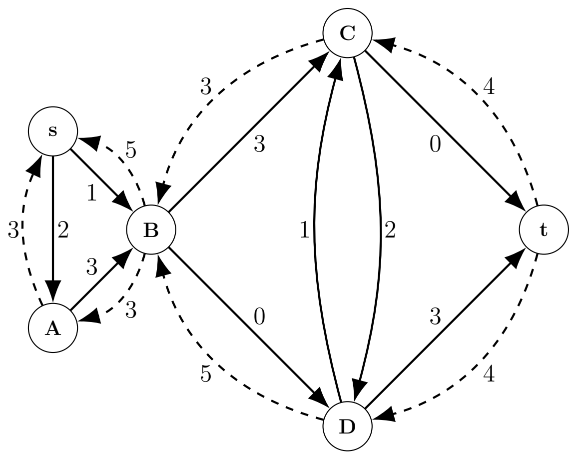
Question 6: We prove that G is the complete graph by contradiction. Assume G is not the complete graph, then there exists two distinct vertices $x, y \in V(G)$ not connected by an edge in G, i.e. $xy \notin E(G)$. By the property of G, there exists a proper $(\chi(G)-2)$-coloring of $G - x - y$. We denote this coloring c and the colors it uses $\{1, \dots, \chi(G)-2\}$. Now, when $G - x - y$ is colored by c, the neighbors of x, which do not include y, are colored with the set of colors $\{1, \dots, \chi(G)-2\}$. Thus c can be extended to a proper coloring including x by assigning it the color $\chi(G)-1$. Similarly, c can be extended to y by assigning it the color $\chi(G)-1$. Hence, we have designed a proper $(\chi(G)-1)$-coloring of G. This is a contradiction.

Question 7: Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two perfect matchings of a tree T. Consider the subgraph G of T with $V(G) = V(T)$ and $E(G) = \mathcal{M}_1 \Delta \mathcal{M}_2$. Then every vertex $v \in V(G)$ has degree 0 or 2. So the graph G is a disjoint union of isolated vertices and cycles. However, a tree is cycle-free. Therefore every vertex in G has degree 0, which implies that $\mathcal{M}_1 = \mathcal{M}_2$.

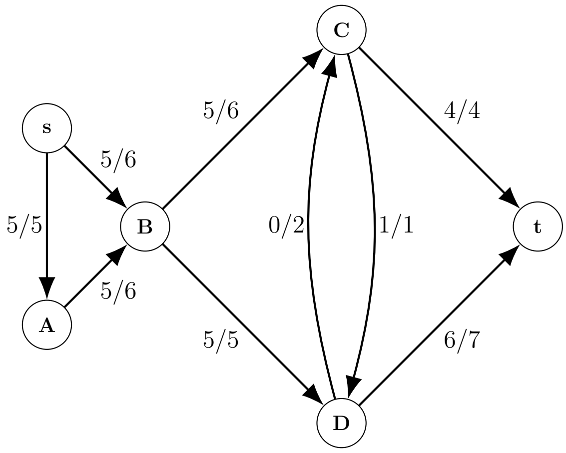
Question 8: To apply, if possible, one iteration of the Ford-Fulkerson algorithm to the network with the existing flow, we need to find an augmenting path, i.e. a s,t-path using only edges with strictly positive residual capacities.

Indeed there exist two: $\mathcal{P}_1 = s, A, B, C, D, t$ and $\mathcal{P}_2 = s, B, C, D, t$; they can be found using BFS algorithm. Now, using $\mathcal{P}_1$, the maximum residual capacity among all edges is $\delta = 2$, for sA and CD. We can thus increase by 2 the value of the flow along $\mathcal{P}_1$ for this iteration. The resulting flow is shown below on the figure on the left; the associated residual capacities are shown on the figure on the right. Note that there is no augmenting path in this new residual graph, therefore the Ford-Fulkerson algorithm terminates. The flow obtained is maximum, of value 10, and the associated minimum cut is $\{s, A, B, C\}$, $\{D, t\}$.

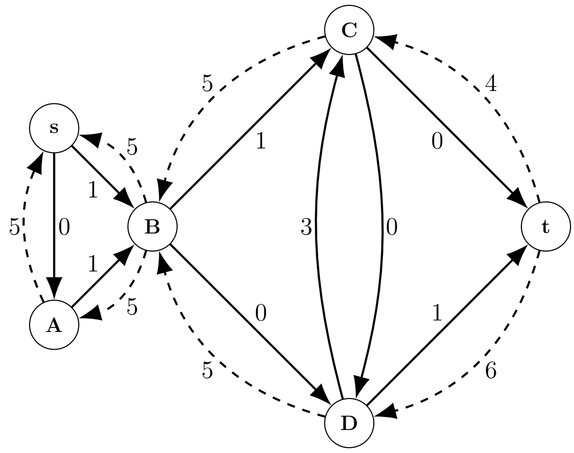
Note that if we select the path $\mathcal{P}_2$ instead, the flow can be increased by 1 only and another iteration is required using $\mathcal{P}_1$, which remains an augmenting path, for the algorimth to terminates.



Initial residual graph and capacities



New flow



New residual graph and capacities

Question 9:

(a) Considering all possible edges, there are 3 different paths of length 2 among any triplet of vertices. Now, for such path to exist in the random graph $G \in \mathcal{G}(n,p)$, at least 2 edges among the 3 connecting any 3 vertices must exist. This happends with probability p^2 because edges exist in G with independent probabilities, all equal to p. Thus, the expected number of paths of length 2 in G is: $3\binom{n}{3}p^2$.

(b) For each set A of s vertices, let X_A be the indicator random variable that A forms a red clique in K_n colored according to the random process. Then $X = \sum_A X_A$ is the random variable that counts the number of red s-cliques in this randomly colored graph. Similarly, for each set B of t vertices, let Y_B be the indicator random variable that B forms a blue clique. Then $Y = \sum_B Y_B$ is the random variable that counts the number of blue t-cliques. Then We have:

$$\mathbb{E}[X] = \sum_{\substack{A \subset V(K_n) \\ |A|=s}} \mathbb{E}[X_A] = \binom{n}{s} p^{\binom{s}{2}}, \quad \mathbb{E}[Y] = \sum_{\substack{B \subset V(K_n) \\ |B|=t}} \mathbb{E}[Y_B] = \binom{n}{t}(1-p)^{\binom{t}{2}}.$$

(c) From question (b), we now know that the number of expected number of red s-cliques and blue t-cliques in K_n randomly 2-edge-colored is:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = \binom{n}{s} p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}}$$

Thus there exists an edge-coloring c, such that the total number of red s-cliques and blue t-cliques is at most $\mathbb{E}[X+Y]$. Considering such a coloring of the edges of K_n, delete one vertex for each red s-clique and blue t-clique. We then get a complete graph with $n - \binom{n}{s}p^{\binom{s}{2}} - \binom{n}{t}(1-p)^{\binom{t}{2}}$ vertices, for which the coloring c contains no red K_s or blue K_t.

Question 10: Let v be an eigenvector of A_G with eigenvalue λ and suppose its ith coordinate v_i is the largest in absolute value (hence $|v_i| > 0$). We know that the ith coordinate of $A_G \cdot v$ is λv_i. On the other hand, this coordinate is equal to the product of the ith row of A_G and v.

As G is d-regular, the ith row contains d entries of value 1, say at coordinates $J \subset \{1, \ldots, n\}$, all others being 0. Then we have:

$$|\lambda||v_i| = |\lambda v_i| = |(A_G \cdot v)_i| = |\sum_{j \in J} v_j| \leq \sum_{j \in J} |v_j| \leq d|v_i|$$

Hence $|\lambda| \leq d$, as requested.

Bonus question: To show that $\text{ex}(n, P_{k+1}) \geq \frac{n(k-1)}{2}$, we can simply consider the n/k disjoint unions of k-cliques: it contains exactly n vertices and many paths of length k.
Showing that $\text{ex}(n, P_{k+1}) \leq \frac{n(k-1)}{2}$ is equivalent to show that any graph G with $|E(G)| > \frac{n(k-1)}{2}$ contains a path P_{k+1}. We prove it by induction on n. Suppose that it is true for graphs with at most $n-1$ vertices.

1. Suppose that G is connected. If for every $v \in G$ we have $d(v) > \frac{k-1}{2}$, then for any $u, v \in G$, $d(u) + d(v) > k-1$, and by the lemma from question 4 there must be a P_{k+1} contained in G. Otherwise, consider the subgraph G' by removing a vertex of G with degree smaller or equal to $\frac{k-1}{2}$. Then $|E(G')| > \frac{(n-1)(k-1)}{2}$, which by induction implies that G' contains P_{k+1}.

2. Suppose now that G is not connected. Consider its connected component H with the largest ratio $\frac{|E(H)|}{|V(H)|}$ which is strictly larger than $\frac{k-1}{2}$, or equivalently $|E(H)| > \frac{|H|(k-1)}{2}$. Then by induction it implies that H contains P_{k+1}.