Graph Theory
MATH - 360

EPFL - Fall Semester 2024/2025
January 18th, 2025

Final exam

Name:
SCIPER:

Question:	1	2	3	4	5	6	7	8	9	10	BONUS	Total (/100):
Score:												

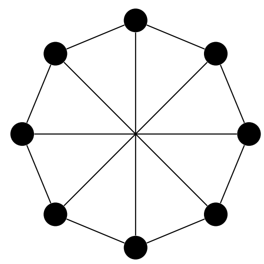
Instructions

- **Do not turn this page before the exam starts and you are allowed to do so.**
- The exam sheet contains 10 questions; it is evaluated on a total of 100 points. One *bonus* question counts for additional 10 points.
- Before the exam starts:
  - Check that you are at your assigned seat; see the listing at the entrance.
  - Put your SCIPER card or your ID/passport on your desk.
  - Take only necessary writing equipment (no blank page or draft paper).
- During the exam:
  - No exit is allowed during the first 30 min and last 15 min of the exam.
  - Leaving the room is allowed only accompanied by one of the supervisors.
  - No document or electronic device are allowed.
  - No question regarding the content of the exam will be answered. If you spot what you think is an error, mention it in your copy. Appropriate measures will be put in place during grading if there were actually an error.
- At the end of the exam / upon finishing:
  - Stop writting as soon as you are requested to and remain seated. If you finish earlier, remain seated and call for one of the supervisors.
  - Check that your name and SCIPER number is written on all sheets of your copy.
  - You are asked to sign the presence sheet while your copy is collected. You must hand your copy and the exam sheet at the end of the exam.

DO NOT OPEN BEFORE THE BEGINNING OF THE EXAM

Reminder of notations: Let $G = (V, E)$ be a graph. For $W \subseteq V$, $G - W$ denotes the graph with vertex set $V \setminus W$ and edge set $E \setminus \{e = uv \in E : \{u, v\} \cap W \neq \emptyset\}$. For $F \subseteq E$, $G \setminus F$ denotes the graph with vertex set V and edge set $E \setminus F$.

Question 1 [7 points]: The graph shown on the right is called the Wagner graph; beware that there is no vertex at the very center. Without any justification, answer the following questions: what is the diameter of the graph? Its girth? Its independence number? Its chromatic number? Is it planar? Eulerian? Hamiltonian?



Question 2 [7 points]: Let G be a connected graph on $n \geq 2$ vertices. Prove that there exists a vertex such that, after removing it from G, the resulting graph is still connected.

Question 3 [10 points]: Consider an arbitrary 2-edge-coloring red and blue of the complete graph K_n, with $n \geq 3$. Prove that there is a Hamiltonian cycle that is the union of at most two monochromatic paths.

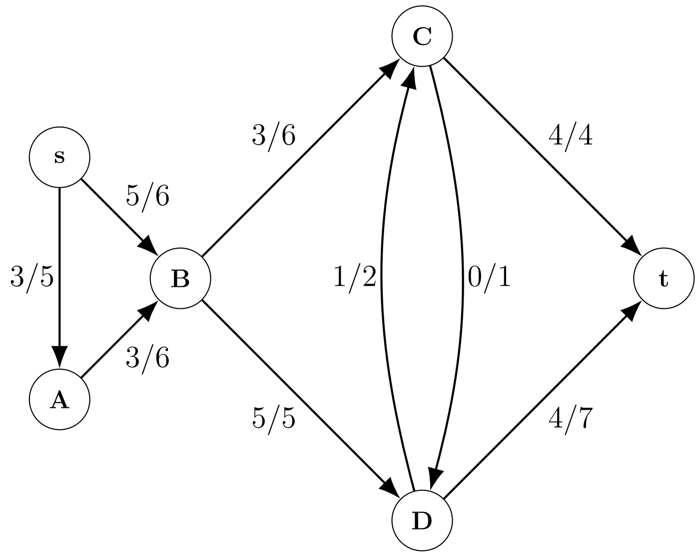
Question 4 [12 points]: Recall Ore's theorem regarding Hamiltonicity: *Let G be a graph on $n \geq 3$ vertices. If $d(u) + d(v) \geq n$ for any non-adjacent vertices u and v, then G contains a Hamilton cycle.* Use the same proof technique to prove the following lemma: *Let G be a connected graph on $n \geq 3$ vertices and $k < n$ an integer. If $d(u) + d(v) \geq k$ for any non-adjacent vertices u and v, then G contains a path of length k.*

Question 5 [6 points]: Let G be a planar graph with fewer than 12 vertices. Show that G has a vertex of degree at most 4.

Question 6 [10 points]: Let G be a graph such that $\chi(G - \{x, y\}) = \chi(G) - 2$ for all pairs of distinct vertices $x, y \in V(G)$. Prove that G is the complete graph.

Question 7 [11 points]: Show that a tree has at most one perfect matching.

Question 8 [8 points]: The figure below shows a network with a source vertex s, a sink vertex t and an existing flow. The value of the flow f and capacity c on each directed edge is represented under the form f/c. On your copy, draw a figure representing the residual graph with residual capacities of each edge associated with this flow. Apply one iteration of the Ford-Fulkerson algorithm on this network starting from this flow. Draw two other figures representing the new values of the flow and residual capacities for each edge at the end of this single iteration. Is it the last iteration of the algorithm? (no justification required)



Question 9 [19 points]: Let $n \in \mathbb{N}^*$ and $p \in [0, 1]$.

(a) [6 points] Consider G the graph on n vertices obtained by the random process of independently connecting each pair of distinct vertices $u, v \in V(G)$ with probability p. What is the expected number of paths of length 2 in G?

(b) [5 points] Consider the random 2-edge-coloring of K_n where each edge is colored in red with probability p and in blue with probability $1 - p$ independently from other edges. Given $s, t \in \mathbb{N}^*$, what is the expected number of red s-cliques in K_n, and what is the expected number of blue t-cliques in K_n?

(c) [8 points] Using question (b), prove the following inequality:

$$R(s,t) \geq n - \binom{n}{s} p^{\binom{s}{2}} - \binom{n}{t} (1-p)^{\binom{t}{2}}$$

Question 10 [10 points]: Let G be a d-regular graph. Prove that if λ is an eigenvalue of A_G, the adjacency matrix of G, then $|\lambda| \leq d$.

Bonus question [10 points]: Let k, n be positive integers such that k divides n. Using the lemma from question 4, prove that $\text{ex}(n, P_{k+1}) = \frac{n(k-1)}{2}$, where P_{k+1} denotes the path of length k. You can fix k and use induction on n. Beware that the lemma requires G to be connected.