Problem Sheet 11

November 29, 2024

Question 1

Let $u :]0,1[^2 \to \mathbb{R}$ be such that :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,y,t) + \dfrac{\partial u}{\partial x}(x,y,t) = 0, & (x,y) \in]0,1[^2, \quad t > 0; \\ u(x,y,0) = 1, & (x,y) \in]0,1[^2; \\ u(0,y,t) = 0, & y \in]0,1[\quad t > 0. \end{cases} \tag{1}$$

(a) What is the exact solution at $t = 0.5$?

(b) Fill the **teansp2d.m** file, run the code with $N = 99, 199, 399$ and $M = 60, 120, 240$ to check (eye) convergence.

(c) What happens whe $N = 99$ and $M = 50$?

Question 2

Graded Exercise for Group 1

Given $u_0, \sigma_0 : [0,1] \to \mathbb{R}$, let $u, \sigma : [0,1] \times [0,T] \to \mathbb{R}$ be such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial \sigma}{\partial x}(x,t), & 0 \leqslant x \leqslant 1, \ 0 \leqslant t \leqslant T, \\ \dfrac{\partial \sigma}{\partial t}(x,t) = c^2 \dfrac{\partial u}{\partial x}(x,t), & 0 \leqslant x \leqslant 1, \ 0 \leqslant t \leqslant T, \\ u(0,t) = 0, & 0 \leqslant t \leqslant T, \\ u(1,t) = 0, & 0 \leqslant t \leqslant T, \\ u(x,0) = u_0(x), & 0 \leqslant x \leqslant 1, \\ \sigma(x,0) = \sigma_0(x), & 0 \leqslant x \leqslant 1. \end{cases}$$

(a) Verify that u satisfies the wave equation with $f(x,t) = 0$ and $\dfrac{\partial u}{\partial t}(x,0) = \sigma_0'(x)$.

(b) Let $h = \dfrac{1}{N+1}$, $x_i = ih$, $i = 1, \dots, N+1$, $x_{i+\frac{1}{2}} = (i + \frac{1}{2})h$, $i = 0, \dots, N$, $\tau = \dfrac{T}{M}$, $t_n = n\tau$, $n = 0, \dots, M$. Let u_i^n and $\sigma_{i+\frac{1}{2}}$ be approximations of $u(x_i, t_n)$ and $\sigma(x_{i+\frac{1}{2}}, t_n)$. We consider the following scheme (Stormer-Verlet, staggered) :
For $n = 0, 1, \dots, M-1$, set $u_0^{n+1} = u_{N+1}^{n+1} = 0$ and

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{1}{2} \frac{\sigma_{i+\frac{1}{2}}^n - \sigma_{i-\frac{1}{2}}^n}{h} + \frac{1}{2} \frac{\sigma_{i+\frac{1}{2}}^{n+1} - \sigma_{i-\frac{1}{2}}^{n+1}}{h}, \ i = 1, \dots, N,$$
$$\frac{\sigma_{i+\frac{1}{2}}^{n+1} - \sigma_{i+\frac{1}{2}}^n}{\tau} = c^2 \frac{1}{2} \frac{u_{i+1}^{n+1} - u_i^{n+1}}{h} + c^2 \frac{1}{2} \frac{u_{i+1}^n - u_i^n}{h}, \ i = 0, 1, \dots, N.$$

Verify that the u_i^n satisfy the implicit Newmark scheme from the lecture.