# Problem Sheet 10

November 18, 2024

## Question 1

Consider the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, & 0 < x < 1,\ t > 0,\ c_0 > 0, \\ u(0,t) = 0, & t > 0, \\ u(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1, \end{cases} \tag{1}$$

and the Crank-Nicholson scheme :

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\tau} + A\frac{\vec{u}^{n+1} + \vec{u}^n}{2} = 0.$$

**(a)** Prove that if $\tau \leqslant h^2$, then $||\vec{u}^{n+1}||_\infty \leqslant ||\vec{u}^n||_\infty$.

**(b)** Prove that, for every $\tau > 0$, $||\vec{u}^{n+1}||_2 \leqslant ||\vec{u}^n||_2$.

## Question 2

Let $T > 0$, $f : [0,1] \times [0,T] \to \mathbb{R}$, $u_0 : [0,1] \to \mathbb{R}$ and $f : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), & 0 < x < 1,\ 0 < t \leqslant T; \\ u(0,t) = 0, & 0 < t \leqslant T; \\ u(1,t) = 0, & 0 < t \leqslant T; \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases} \tag{2}$$

Prove that if $f(x,t) \geqslant 0\ \forall (x,t) \in [0,1] \times [0,T]$ and $u_0(x) \geqslant 0\ \forall x \in [0,1]$, then $u(x,t) \geqslant 0\ \forall (x,t) \in [0,1] \times [0,T]$. Indication : Write $u$ as $u = u_+ - u_-$, where $u_+ := \max\{u, 0\}$ and $u_- := \max\{-u, 0\}$.

## Question 3

Implement the upwind and Lax-Wendroff schemes for the transport equation :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + c_0\dfrac{\partial u}{\partial x}(x,t) = 0, \qquad 0 < x < 1, \qquad 0 < t < 0.48, \qquad c_0 = \sqrt{2}; \\ u(0,t) = 0, \qquad t > 0; \\ u(x,0) = e^{-1000(x-0.2)^2}, \qquad 0 < x < 1. \end{cases} \tag{3}$$

Compare both schemes at time 0.48 when $h = 0.01$, $\Delta t = 0.006$ and check convergence.

## Question 4

## Graded Exercise for Group 3

Consider the Lax-Wendroff scheme for the transport equation :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + c_0\dfrac{\partial u}{\partial x}(x,t) = 0, & x \in \mathbb{R},\ t > 0,\ c_0 > 0; \\ u(x,0) = e^{imx} & x \in \mathbb{R},\ m \in \mathbb{N}^*. \end{cases} \tag{4}$$

**(a)** Check that $u_j^n = u_j^0 \gamma^n$, with $\gamma = 1 - \alpha^2(1 - \cos(mh)) - i\alpha\sin(mh)$, $\alpha = \dfrac{c_0\tau}{h}$.

**(b)** Check that $|\gamma|^2 = 1 + \alpha^2(\alpha^2 - 1)(1 - \cos(mh))^2$, so that $|\gamma| \leqslant 1$ when $\alpha \leqslant 1$.