Time Series Exercise Sheet 13

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Exercise 13.1

Assume the generation of r_t is governed by an ARCH (AutoRegressive Conditionally Heteroscedastic) model. This particular instance takes the form for β_j with $j = 0, 1, 2$ non-negative constants and ε_t a zero-mean process:

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \beta_0 + \beta_1 r_{t-1}^2 + \beta_2 r_{t-2}^2, \end{aligned}$$

and ε_t is assumed to have unit variance for identifiability. Let F_t denote all the information available at time point t

(a) Calculate $\mathbb{E}[r_t]$ using the law of iterated expectation, conditioning on F_{t-1}.

(b) Assuming r_t is stationary, calculate the marginal variance of r_t, quoting any laws of probability you may need.

(c) Determine necessary conditions for r_t to be stationary in terms of β_j with $j = 0, 1, 2$.

Solution 13.1

1. Since ϵ_t is independent of F_{t-1},

$$\begin{aligned} \mathrm{E}(r_t \mid F_{t-1}) &= \mathrm{E}(\sigma_t \epsilon_t \mid F_{t-1}) \\ &= \sigma_t \mathrm{E}(\epsilon_t) \\ &= 0 \\ \mathrm{E}(r_t) &= \mathrm{E}\{\mathrm{E}(r_t \mid F_{t-1})\} \\ &= 0 \end{aligned}$$

2. Since ϵ_t is independent of F_{t-1}, by the law of total variance, it holds that

$$\begin{aligned} \mathrm{Var}(r_t \mid F_{t-1}) &= \mathrm{Var}(\sigma_t \epsilon_t \mid F_{t-1}) \\ &= \left(\beta_0 + \beta_1 r_{t-1}^2 + \beta_2 r_{t-2}^2\right) \mathrm{Var}(\epsilon_t) \\ &= \beta_0 + \beta_1 r_{t-1}^2 + \beta_2 r_{t-2}^2 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(r_t) &= \mathbb{E}[\mathrm{Var}(r_t \mid F_{t-1})] + \mathrm{Var}(\mathbb{E}[r_t \mid F_{t-1}]) \\ &= \mathbb{E}\left[\beta_0 + \beta_1 r_{t-1}^2 + \beta_2 r_{t-2}^2\right] + 0 \\ &= \beta_0 + \beta_1 \mathrm{Var}(r_{t-1}) + \beta_2 \mathrm{Var}(r_{t-2}) \end{aligned}$$

since $\mathbb{E}[r_t] = 0$ by (a). Then, $\mathrm{Var}(r_t) = \mathrm{Var}(r_{t-1}) = \ldots = \mathrm{Var}(r_1)$, it holds that

$$\begin{aligned} (1 - \beta_1 - \beta_2) \mathrm{Var}(r_t) &= \beta_0 \\ \Rightarrow \mathrm{Var}(r_t) &= \frac{\beta_0}{1 - \beta_1 - \beta_2} \end{aligned}$$

if $\beta_1 + \beta_2 < 1$.

3. To be stationary, $\beta_1 + \beta_2 < 1$, so the variance of r_t in (b) is positive.

Exercise 13.2

The generalised autoregressive conditionally heteroscedastic (GARCH) class of models is given by

$$r_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \sum_{j=1}^{m} \alpha_j r_{t-j}^2 + \sum_{j=1}^{r} \beta_j \sigma_{t-j}^2, \quad \varepsilon_t \overset{iid}{\sim} \mathcal{N}(0,1) \tag{1}$$

and labelled GARCH(m, r). To avoid negative variances, we take $\alpha_j, \beta_j \geq 0$.

A GARCH$(m, 0)$ is an ARCH(m). Prove that a GARCH model has zero mean, and satisfies: $\text{Corr}\left(r_{t+\tau}, r_t\right) = 0$.

Remember that F_τ denote the entire history of the process $\{r_t\}$ up to time τ.

Solution 13.2

Conditional on the history, σ_t is constant, so

$$\begin{aligned}
\mathbb{E}\left[r_t\right] &= \mathbb{E}\left[\mathbb{E}\left[r_t \mid F_{t-1}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\sigma_t \varepsilon_t \mid F_{t-1}\right]\right] \\
&= \mathbb{E}\left[\sigma_t \mathbb{E}\left[\varepsilon_t \mid F_{t-1}\right]\right] \\
&= 0
\end{aligned}$$

The two other results follow from the same argument as the lectures.

Exercise 13.3

Show that for an ARCH(1) with parameters $0 < \alpha_0,\ 0 < \alpha_1 < 1/3$ we have $\text{Corr}\left(r_{+\tau}^2, r_t^2\right) = \alpha_1^{|\tau|}$

Solution 13.3

We start with the covariance, assuming that $\tau > 0$.

$$\begin{aligned}
\text{Cov}\left(r_{t+\tau}^2, r_t^2\right) &= \mathbb{E}\left[r_{t+\tau}^2 r_t^2\right] - \mathbb{E}\left[r_{t+\tau}^2\right]\mathbb{E}\left[r_t^2\right] \\
&= \mathbb{E}\left[r_{t+\tau}^2 r_t^2\right] - \mathbb{E}\left[r_t^2\right]^2 \\
&= \mathbb{E}\left[\mathbb{E}\left[r_{t+\tau}^2 r_t^2 \mid F_{t+\tau-1}\right]\right] - \mathbb{E}\left[r_t^2\right]^2 \\
&= \mathbb{E}\left[r_t^2 \sigma_{t+\tau}^2 \mathbb{E}\left[\varepsilon_{t+\tau}^2 \mid F_{t+\tau-1}\right]\right] - \mathbb{E}\left[r_t^2\right]^2 \\
&= \mathbb{E}\left[r_t^2(\alpha_0 + \alpha_1 r_{t+\tau-1}^2)\right] - \mathbb{E}\left[r_t^2\right]^2 \\
&= \alpha_0 \mathbb{E}\left[r_t^2\right] + \alpha_1 \mathbb{E}\left[r_{t+\tau-1}^2 r_t^2\right] - \mathbb{E}\left[r_t^2\right]^2 \\
&= \alpha_1 \,\mathbb{C}\text{ov}\, r_{t+\tau-1}^2, r_t^2 + \alpha_0 \mathbb{E}\left[r_t^2\right] - (1-\alpha_1)\mathbb{E}\left[r_t^2\right]^2
\end{aligned}$$

Recall from the notes that

$$\mathbb{E}\left[r_t^2\right] = \text{Var}\left(r_t\right) = \frac{\alpha_0}{1-\alpha_1}.$$

Therefore

$$\begin{aligned}
\text{Cov}\left(r_{t+\tau}^2, r_t^2\right) &= \alpha_1 \,\mathbb{C}\text{ov}\, r_{t+\tau-1}^2, r_t^2 + \frac{\alpha_0^2}{1-\alpha_1} - (1-\alpha_1)\left(\frac{\alpha_0}{1-\alpha_1}\right)^2 \\
&= \alpha_1 \,\mathbb{C}\text{ov}\, r_{t+\tau-1}^2, r_t^2.
\end{aligned}$$

Therefore by a recursive argument we have

$$\text{Cov}\left(r_{t+\tau}^2, r_t^2\right) = \alpha_1^\tau \,\text{Var}\left(r_t^2\right).$$

Now by symmetry of the autocovariance and definition of correlation we have

$$\text{Corr}\left(r_{+\tau}^2, r_t^2\right) = \alpha_1^{|\tau|}.$$