Time Series Exercise Sheet 12

Sofia Olhede

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Exercise 12.1

The model fitted to $z_1, \ldots, z_{100}$ if the model

$$Z_t = -0.4Z_{t-1} + \epsilon_t$$

where ϵ_t is unit variance Gaussian white noise. We have estimate the ACF and the PACF for the first 12 lags in Table 1.

Table 1: Autocorrelation and Partial Autocorrelation Estimates

lag	1	2	3	4	5	6	7	8	9	10	11	12
ACF	0.799	0.412	0.025	-0.228	-0.316	-0.287	-0.198	-0.111	-0.056	-0.009	0.048	0.133
PACF	0.799	-0.625	-0.044	0.038	-0.020	-0.077	-0.077	-0.061	-0.042	0.089	0.052	0.125

Are these values consistent with the residuals being white? Please explain why/why not?

Solution 12.1

In order to check if the order of the fitted AR component is appropriate we have to look at the partial ACF. We recall from the lecture that for an AR(p) process, the PACF at lag k is zero, namely $\alpha_\tau = 0$ for $\tau > p$.

What we see, however, is that $|\alpha_2| = 0.625 > 0$, and in particular is well beyond the confidence interval of $1.96/\sqrt{N} = 0.196$ for $N = 100$. This suggests that the order of the fitted AR model is not appropriate, and hence also the residuals would not be white noise in this case.

Exercise 12.2

Show when AIC and AICC start to become the same. Explain your rationale.

Solution 12.2

We first write the formulas of both:

$$\text{AIC}(\boldsymbol{\theta}) = -2\ln L(\boldsymbol{\theta}|\boldsymbol{y}) + 2k, \qquad \text{AICC}(\boldsymbol{\theta}) = -2\ln L(\boldsymbol{\theta}|\boldsymbol{y}) + 2k\frac{N}{N-k-1}. \tag{1}$$

Recall that the term $N/(N-k-1)$ is added in order to penalize for the number of parameters that are fitted (avoid overfitting) to the model when the sample size N is small. Clearly, however, as $N \to \infty$ (in practice as the sample size grows larger) we see that $N/(N-k-1) \to 1$, and thus, AICC$\to$AIC as $N \to \infty$.

Exercise 12.3

Determine the form of AICC when you assume that m coefficients of the ARMA(p,q) model are zero.

Solution 12.3

In case we assume that only $p+q-m$ parameters, out of $p+q$, are non-zero, this means that we only estimate those $p+q-m$ parameters, while imposing restrictions on the remaining (zero) parameters. Therefore, in this case there is no need to penalize by choosing a larger k in the AICC formula above. We would obtain

$$\mathrm{AICC}(\boldsymbol{\theta}) = -2 \ln L(\boldsymbol{\theta}|\boldsymbol{y}) + 2(p+q+1-m)\frac{N}{N-(p+q+1-m)-1}. \tag{2}$$

On the other hand, if we fit the full ARMA(p, q) model without imposing any restrictions on the coefficients, and if it turns out that a number of the estimated coefficients are close to zero, then we still need to take account of this by selecting $k = p + q + 1$ instead of $(p + q + 1 - m)$ above.

Exercise 12.4

Prove that, for an AR(p), we have

$$\alpha_\tau = 0, \qquad \forall \tau > p.$$

Solution 12.4

Since the partial autocorrelation is the last coefficient in the best linear predictor, the result follows trivially from the form of the best linear predictor of an AR(p).

Exercise 12.5

For the 6 time series shown in Figure 1, identify if any of the following models could be appropriate

1. White noise,
2. MA(q),
3. AR(p),
4. ARMA(p, q).

Give you reasoning.

Solution 12.5

The true underlying processes are as follows:

1. ARMA$(1, 1)$,
2. MA(1),
3. AR(1),
4. white noise,
5. not stationary,
6. AR(1).

However, the important point is that you make a sensible argument for your choice in each case. Example arguments for some reasonable models are given below:

1. Looking at the time series, it looks like the mean and variance are not changing over time. Furthermore, both the ACF and PACF decay in magnitude and do not sharply drop off after some point. Therefore, we might suggest an ARMA model.
2. Again the time series seems to be stationary. We see a sharp drop in the ACF after lag 1, and a decay in the PACF. This suggests an MA(1) model is appropriate.
3. The process looks to be stationary, the ACF decays in magnitude as lag increases, but the PACF drops after lag 1. Therefore an AR(1) model is likely appropriate.
4. The process seems stationary, and both the PACF and ACF are not significant for non-zero lags. Therefore a white noise model is likely to be appropriate.
5. This process has very strong lag 1 PACF, and the ACF decays very slowly. In conjunction, the time series looks as though it could have a changing variance, so this is likely not to be stationary. This should be explored further, but a stationary model is likely not appropriate.

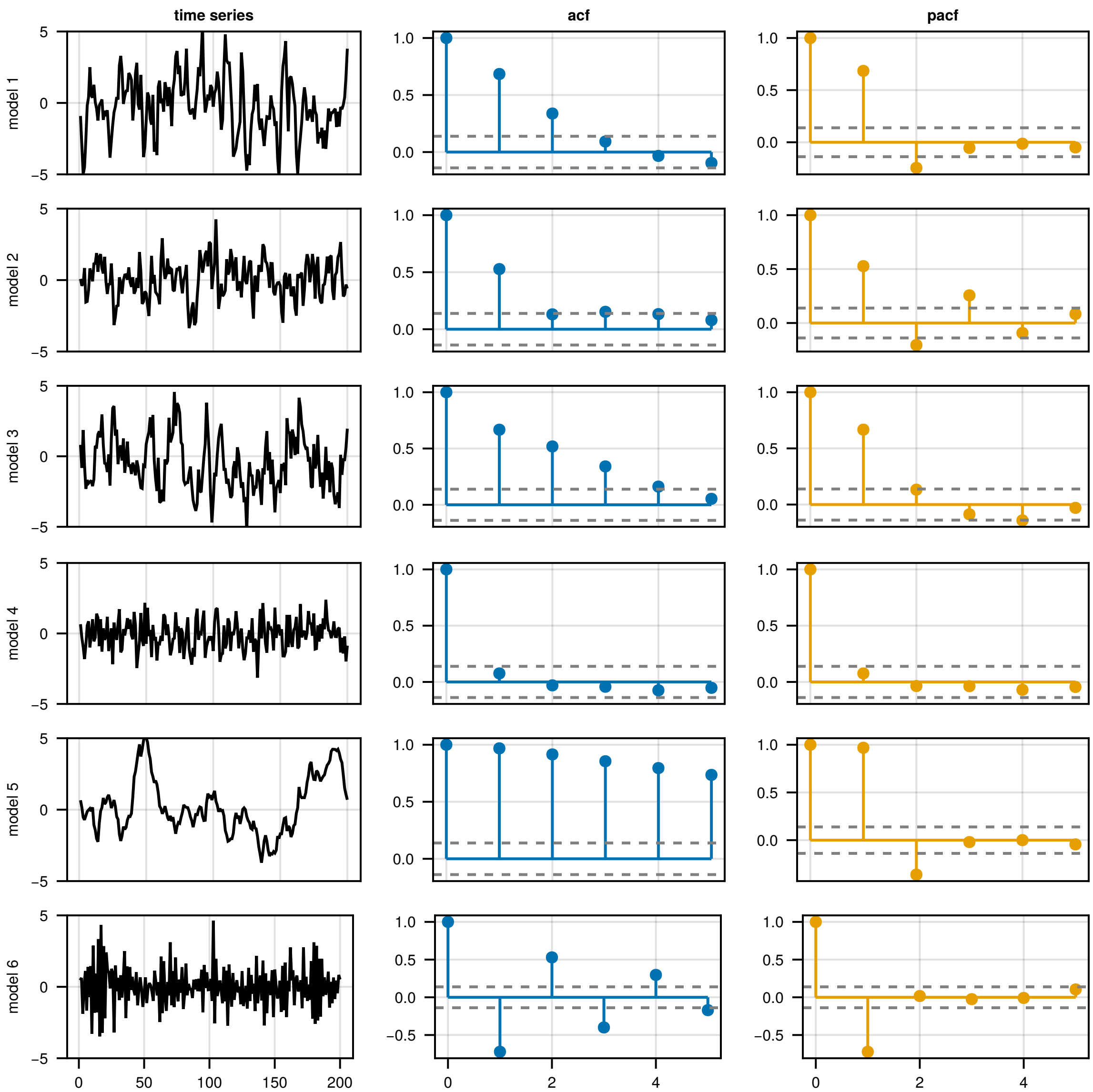

Figure 1: Time series and their sample acf and pacfs

6. This process looks stationary, the ACF decays in magnitude, but the PACF drops to non-significant after lag 1. Therefore an AR(1) model may be appropriate.

Note that in some of these cases, one could argue for a different model. For example, one could possibly argue for an AR(2) in the first example, or possibly an ARMA model in the third example. One could also argue for an AR(1) with very high correlation in the fifth example. The purpose of this exercise is to practice making such arguments, and to realise that model selection is not trivial! There are other forms of model selection one can use to choose an appropriate model, some of which will be seen later in the course. But looking at the time series, the ACF and the PACF are useful sanity checks in any case.