Time Series Exercise Sheet 11

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Exercise 11.1

Consider two LTI filters L_1 and L_2, and let $\alpha \in \mathbb{C}$ then the filter

$$L = \alpha L_1 + L_2 \tag{1}$$

is also an LTI filter.

Solution 11.1

Let $\{X_t\}$, $\{Y_t\}$ be sequences, and $\beta \in \mathbb{C}$, then

$$\begin{aligned} L\left[\beta \{X_t\} + \{Y_t\}\right] &= \alpha L_1\left[\beta \{X_t\} + \{Y_t\}\right] + L_2\left[\beta \{X_t\} + \{Y_t\}\right] \\ &= \alpha\beta L_1\left[\{X_t\}\right] + \alpha L_1\left[\{Y_t\}\right] + \beta L_2\left[\{X_t\}\right] + L_2\left[\{Y_t\}\right] \qquad \text{(linearity of LTI filters)} \\ &= \beta L\left[\{X_t\}\right] + L\left[\{Y_t\}\right]. \end{aligned}$$

Now for time invariance,

$$\begin{aligned} L\left[\mathrm{B}\left[\{X_t\}\right]\right] &= \alpha L_1\left[\mathrm{B}\left[\{X_t\}\right]\right] + L_2\left[\mathrm{B}\left[\{X_t\}\right]\right] \\ &= \mathrm{B}\left[\alpha L_1\left[\{X_t\}\right]\right] + \mathrm{B}\left[L_2\left[\{X_t\}\right]\right] \qquad \text{(linearity of LTI filters)} \\ &= \mathrm{B}\left[L\left[\{X_t\}\right]\right]. \end{aligned}$$

Exercise 11.2

Consider two LTI filters L_1 and L_2. The filter $L = L_1 L_2$, i.e. so that

$$L\left[\{X_t\}\right] = L_1\left[L_2\left[\{X_t\}\right]\right] \tag{2}$$

is also an LTI filter.

Solution 11.2

Let $\{X_t\}$, $\{Y_t\}$ be sequences, and $\alpha \in \mathbb{C}$, then

$$\begin{aligned} L\left[\alpha \{X_t\} + \{Y_t\}\right] &= L_1\left[L_2\left[\alpha \{X_t\} + \{Y_t\}\right]\right] \\ &= L_1\left[\alpha L_2\left[\{X_t\}\right] + L_2\left[\{Y_t\}\right]\right] \\ &= \alpha L_1\left[L_2\left[\{X_t\}\right]\right] + L_1\left[L_2\left[\{Y_t\}\right]\right] \\ &= \alpha L\left[\{X_t\}\right] + L\left[\{Y_t\}\right]. \end{aligned}$$

Now for time invariance

$$\begin{aligned} L\left[\mathrm{B}\left[\{X_t\}\right]\right] &= L_1\left[L_2\left[\mathrm{B}\left[\{X_t\}\right]\right]\right] \\ &= L_1\left[\mathrm{B}\left[L_2\left[\{X_t\}\right]\right]\right] \\ &= \mathrm{B}\left[L_1\left[L_2\left[\{X_t\}\right]\right]\right]. \end{aligned}$$

Exercise 11.3

A digital filter L is an LTI filter if and only if we can write the filter output as a convolution:

$$L\left[\{X_t\}\right]_u = \Delta \sum_{m \in \mathcal{T}} h_{u-m} X_m \tag{3}$$

for any $u \in \mathcal{T}$.

Solution 11.3

($\Rightarrow$) Assume that L is a linear time invariant filter, then notice that for any $t \in \mathcal{T}$

$$X_t = \sum_{m \in \mathcal{T}} \delta_{t,m} X_m.$$

Therefore

$$\begin{aligned}
L\left[\{X_t\}\right]_u &= L\left[\sum_{m \in \mathcal{T}} \{\delta_{t,m}\} X_m\right]_u \\
&= \sum_{m \in \mathcal{T}} X_m L\left[\{\delta_{t,m}\}\right]_u \\
&= \sum_{m \in \mathcal{T}} X_m \mathrm{B}^{-u/\Delta}\left[L\left[\{\delta_{t,m}\}\right]\right]_0 \\
&= \sum_{m \in \mathcal{T}} X_m L\left[\{\delta_{t,m-u}\}\right]_0 && \text{(time invariance)} \\
&= \Delta \sum_{m \in \mathcal{T}} X_m h_{u-m}.
\end{aligned}$$

($\Leftarrow$) Now assume that there exists some sequence h such that

$$L\left[\{X_t\}\right]_u = \Delta \sum_{m \in \mathcal{T}} h_{u-m} X_m.$$

Then we have for $\alpha \in \mathbb{C}$, and for any $u \in \mathcal{T}$

$$\begin{aligned}
L\left[\alpha\{X_t\} + \{Y_t\}\right]_u &= \Delta \sum_{m \in \mathcal{T}} h_{u-m}(\alpha X_m + Y_m) \\
&= \alpha \Delta \sum_{m \in \mathcal{T}} h_{u-m} X_m + \Delta \sum_{m \in \mathcal{T}} h_{u-m} Y_m \\
&= \alpha L\left[\{X_t\}\right]_u + L\left[\{Y_t\}\right]_u.
\end{aligned}$$

Finally,

$$\begin{aligned}
L\left[\mathrm{B}\left[\{X_t\}\right]\right]_u &= \Delta \sum_{m \in \mathcal{T}} h_{u-m} X_{m-\Delta} \\
&= \Delta \sum_{m' \in \mathcal{T}} h_{u-\Delta-m'} X_{m'} && (m' = m - \Delta) \\
&= L\left[\{X_t\}\right]_{u-\Delta} \\
&= \mathrm{B}\left[L\left[\{X_t\}\right]\right]_u.
\end{aligned}$$

Thus we have L is an LTI filter.

Exercise 11.4

Consider a stationary mean-zero time series $\{X_t\}$, with spectral representation

$$X_t = \int_{-1/2\Delta}^{1/2\Delta} e^{2\pi i f t} \mathrm{d}Z(f). \tag{4}$$

Assume that we observe this time series at the points $T = \{\Delta, \ldots, \Delta n\}$, and we define the tapered discrete Fourier transform by

$$J_h(f) = \sum_{t \in T} h_t X_t e^{-2\pi i f t} \tag{5}$$

where $\|h_t\|_2^2 = 1$ (we assume the mean is known to be zero, so do no mean correction). Show that

$$J_h(f) = \frac{1}{\Delta} \int_{-1/2\Delta}^{1/2\Delta} H(f - f') \mathrm{d}Z(f'). \tag{6}$$

Solution 11.4

We see from the definition that

$$\begin{aligned}
J_h(f) &= \sum_{t \in T} h_t X_t e^{-2\pi i f t} \\
&= \sum_{t \in T} h_t \int_{-1/2\Delta}^{1/2\Delta} e^{2\pi i f' t} \mathrm{d}Z(f') e^{-2\pi i f t} \\
&= \frac{1}{\Delta} \int_{-1/2\Delta}^{1/2\Delta} \Delta \sum_{t \in T} h_t e^{2\pi i (f' - f) t} \mathrm{d}Z(f') \\
&= \frac{1}{\Delta} \int_{-1/2\Delta}^{1/2\Delta} H(f - f') \mathrm{d}Z(f').
\end{aligned}$$

Exercise 11.5

If $\{X_t\}$ is a stationary series, define $\{Y_t\} = (\mathrm{I} - \mathrm{B})[\{X_t\}]$. Is Y_t stationary? If so, what is the spectral density function of $\{Y_t\}$ in terms of the spectral density function of $\{X_t\}$?

Solution 11.5

Note that $(\mathrm{I} - \mathrm{B})$ is an LTI filter with impulse response in ℓ_1, so from Theorem 9.11 $\{Y_t\}$ is stationary. Now we may use Theorem 9.12 to compute the spectral density function. In particular, the transfer function of this filter is given by

$$H(f) = 1 - e^{-2\pi i f \Delta}, \tag{7}$$

and so

$$\begin{aligned}
S_Y(f) &= \left|1 - e^{-2\pi i f \Delta}\right|^2 S_X(f) \\
&= 2(1 - \cos(2\pi f \Delta)) S_X(f) \\
&= 4 \sin^2(\pi f \Delta) S_X(f).
\end{aligned}$$

Exercise 11.6

For this question we fix $\Delta = 1$. In lecture 3, we claimed that AR(p) processes were time reversible. In other words, If $\{X_t\}$ is an AR(p) process, then if $\{Y_t\}$ is such that for all $t \in \mathbb{Z}$, $Y_t = X_{-t}$, then Y_t is an AR(p) process with the same parameters as X_t. Specifically, if X_t had an AR representation

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \epsilon_t \tag{8}$$

then

$$Y_t = \sum_{j=1}^{p} \phi_j Y_{t-j} + \tilde{\nu}_t \tag{9}$$

where $\tilde{\nu}_t$ has the same distribution as ϵ_t. Prove this result.

Solution 11.6

Let $\phi_0 = -1$, and define

$$\tilde{\nu}_t = \sum_{j=0}^{p} \phi_j Y_{t-j}.$$

All that remains is to show that $\tilde{\nu}_t$ is Gaussian white noise with mean zero and variance σ_ϵ^2. Clearly it is Gaussian as it is a linear combination of Gaussians. Furthermore, $\mathbb{E}\left[X_t\right] = 0$, so $\mathbb{E}\left[\tilde{\nu}_t\right] = 0$ by linearity.

Now, we see that for all $\tau \in \mathbb{Z}$

$$\begin{aligned} \gamma_\tau^{(Y)} &= \mathbb{E}\left[Y_t Y_{t+\tau}\right] \\ &= \mathbb{E}\left[X_{-t} X_{-t-\tau}\right] \\ &= \gamma_{-\tau}^{(X)} \\ &= \gamma_\tau^{(X)}. \end{aligned}$$

Therefore $S_Y\left(f\right) = S_X\left(f\right)$ for all $f \in \mathbb{R}$. Finally, we see that $\{\tilde{\nu}_t\}$ results from applying an LTI filter to $\{Y_t\}$. In particular, from Lemma 9.13 and Theorem 9.12, we have

$$\begin{aligned} S_{\tilde{\nu}}\left(f\right) &= \left|\Phi(e^{-2\pi i f \Delta})\right|^2 S_Y\left(f\right) \\ &= \left|\Phi(e^{-2\pi i f \Delta})\right|^2 S_X\left(f\right) \\ &= \Delta\sigma_\epsilon^2 \left|\frac{\Phi(e^{-2\pi i f \Delta})}{\Phi(e^{-2\pi i f \Delta})}\right|^2 \\ &= \Delta\sigma_\epsilon^2 \end{aligned}$$

where the penultimate line follows from Theorem 9.14. Therefore, we know that $\tilde{\nu}$ is white noise with variance σ_ϵ^2.