Time Series lecture 7

Frequency domain time series

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Lecture outline

1. Spectral density (simple case)
   - Continuous vs discrete
2. Spectra for general stationary processes
3. Spectral representation theorem

Spectral density (simple case)

The spectral density function

For simplicity, for the remainder of the course we consider only $\Delta = 1$.

Definition 7.1

Consider a discrete time process $\{X_t\}_{t\in\mathbb{Z}}$, with autocovariance sequence satisfying $\{\gamma_\tau\} \in \ell^1$. We define the spectral density function to be

$$S(f) = \sum_{\tau\in\mathbb{Z}} \gamma_\tau e^{-2\pi i \tau f} \tag{7.1}$$

for all $f \in \mathbb{R}$.

- This is simply the discrete Fourier transform of the autocovariance sequence.
- The subscript indicates that this is the spectral density function of the discrete time process.

The spectral density function (continuous time)

Now consider the continuous time case.

Definition 7.2

Consider a continuous time process $\{X(t)\}_{t\in\mathbb{R}}$, with autocovariance function $\gamma \in L^1$, then the spectral density function is

$$\mathscr{S}(f) = \int_{-\infty}^{\infty} \gamma(\tau)\, e^{-2\pi i\tau f} \mathrm{d}\tau \tag{7.2}$$

for $f \in \mathbb{R}$.

Aliasing

- If we sample a continuous time process at discrete points, then clearly the autocovariance sequence of the sampled process is the autocovariance function of the original process sampled at those same points.
- From the previous lecture, we therefore know that there is an aliasing relation between the spectral density functions of the two respective processes.
- In particular

$$S(f) = \sum_{k \in \mathbb{Z}} \mathscr{S}(f + k). \tag{7.3}$$

- The frequency $\frac{1}{2}$ is called the Nyquist frequency.
- If $\mathscr{S}(f)$ is essentially zero for $|f| \geq \frac{1}{2}$ we can expect a good correspondence between $\mathscr{S}(f)$ and $S(f)$.
- If $\mathscr{S}(f)$ is large for $|f| \geq \frac{1}{2}$ then the correspondence is poor so $S(f)$ tells us nothing about $\mathscr{S}(f)$.

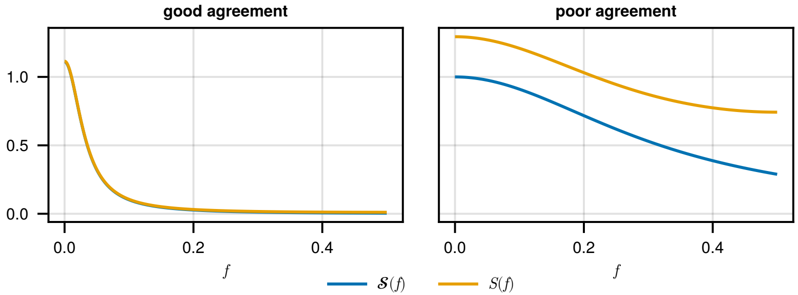

Figure: Aliasing for two different continuous processes

Harmonic processes

Definition 7.3 (Harmonic (Sinusoidal) process)

Consider two independent random variables $A > 0$ and $\Theta \sim \mathrm{Unif}(-\pi, \pi)$. We construct a harmonic process $\{X_t\}$ by setting

$$X_t = A\cos(\nu t + \Theta), \tag{7.4}$$

for all $t \in \mathbb{Z}$ where $\nu \in \mathbb{R}$ determines the frequency of the oscillation.

- We can see that

$$\mathbb{E}[X_t] = 0,$$
$$\mathrm{Cov}(X_{t+\tau}, X_t) = \frac{1}{2}\mathbb{E}\left[A^2\right]\cos(\nu\tau).$$

- This process is stationary, but clearly does not have $\gamma_\tau \in \ell^1$, can we still have a notion of spectra in this case?

Spectra for general stationary processes

Herglotz's theorem

Theorem 7.4 (Herglotz's theorem)

Given a complex valued sequence $\{g_t\}_{t\in\mathbb{Z}}$, then g is non-negative definite if and only if for all $t \in \mathbb{Z}$

$$g_t = \int_{-1/2}^{1/2} e^{2\pi i t f} \, \mathrm{d}G^{(I)}(f) \tag{7.5}$$

where $G^{(I)}(-1/2) = 0$, and $G^{(I)}$ is right continuous, bounded and non-decreasing.

- This can be generalised in a number of ways, but we require only complex valued sequences here.
- Note the properties are essentially the same as a cumulative distribution function.

Integrated spectrum

As a result of Herglotz's theorem, and that the autocovariance sequence is non-negative definite, we have that

$$\gamma_\tau = \int_{-1/2}^{1/2} e^{2\pi i \tau f} \mathrm{d}S^{(I)}(f) \tag{7.6}$$

- $S^{(I)}(\cdot)$ is referred to as the integrated spectrum, or sometimes the spectral distribution function.
- We have the immediate properties:
  (i) $S^{(I)}\left(-\frac{1}{2}\right) = 0$ and $S^{(I)}\left(\frac{1}{2}\right) = \sigma^2$.
  (ii) $f < f' \Rightarrow S^{(I)}(f) \le S^{(I)}(f')$.

Relation to the spectral density function

- If there is a function $S(\cdot)$ such that

$$S^{(I)}(f) = \int_{-1/2}^{f} S(\lambda)\,\mathrm{d}\lambda \tag{7.7}$$

  then $S(\cdot)$ is called the spectral density function.
- This occurs if the autocovariance function is absolutely summable, in which case this is equivalent to the definition of the spectral density function seen earlier in the lecture.

- In various books people use $\omega = 2\pi f$ instead of f. This is called angular frequency rather than frequency. I don't like it, because it causes factors of 2π to fly around all over the place, that are easily missed/forgotten, but it is equivalent.
- Intuitively, $S(f)\,\mathrm{d}f$ is the average contribution over all realisations of the process to the power from components with frequencies in a small interval around f. The variance or "power" of $\{X_t\}$ is

$$\mathrm{Var}(X_t) = \gamma_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i f \cdot 0} S(f)\,\mathrm{d}f.$$

Lebesgue decomposition theorem

Theorem 7.5 (Lebesgue decomposition theorem)

Any integrated spectrum $S^{(I)}(\cdot)$ *can be written as*

$$S^{(I)}(\cdot) = S_1^{(I)}(\cdot) + S_2^{(I)}(\cdot) + S_3^{(I)}(\cdot),$$

where $S_j^{(I)}(\cdot)$ *non-negative, non-decreasing with* $S_j^{(I)}\left(-\frac{1}{2}\right) = 0$ *for* $j = 1, 2, 3$ *and where* $S_1^{(I)}(\cdot)$ *is absolutely continuous,* $S_2^{(I)}(\cdot)$ *is a step function and* $S_3^{(I)}(\cdot)$ *is a continuous singular function.*

- This is actually a more general result, but we state it here for our specific case.
- Now we will turn to the meaning of the different components $S_j^{(I)}(\cdot)$.

Lebesgue decomposition theorem: further detail

(a) $S_1^{(I)}(\cdot)$ is an absolutely continuous function and satisfies

$$S_1^{(I)}(f) = \int_{-1/2}^{f} S(\lambda)\,\mathrm{d}\lambda \tag{7.8}$$

for all $f \in [-1/2, 1/2]$.

(b) $S_2^{(I)}(\cdot)$ is a step function with jumps of size $\{p_l\}$ at the points $\{f_l\}_l$ where f_l are frequencies of pure sinusoids.

(c) $S_3^{(I)}(\cdot)$ is a continuous singular function (pathological and generally of no practical use).

Lebesgue decomposition theorem: some examples

We can then characterise some common scenarios in terms of this decomposition:

(a) This case corresponds to $S_1^{(I)}(f) \geq 0$ and $S_2^{(I)}(f) \equiv 0$. In this case we say that $\{X_t\}$ has a purely continuous spectrum. Note that as $S_1^{(I)}(f)$ is absolutely continuous and non-decreasing (often increasing). Hence its derivative $S(f)$ is absolutely integrable (Titchmarsh, 1960, for example). But note that if $S(f)$ is absolutely integrable $\int_{-1/2}^{1/2} \cos(2\pi f\tau) S(f)\,\mathrm{d}f \to 0$ $\int_{-1/2}^{1/2} \sin(2\pi f\tau) S(f)\,\mathrm{d}f \to 0$ as $\tau \to \infty$. So as $\tau \to \infty$:

$$\gamma_\tau = \int_{-1/2}^{1/2} \cos(2\pi f\tau) S(f)\,\mathrm{d}f + i\int_{-1/2}^{1/2} \sin(2\pi f\tau) S(f)\,\mathrm{d}f \to 0.$$

(b) This case corresponds to $S_1^{(I)}(f) \equiv 0$ and $S_2^{(I)}(f) \geq 0$. Here the integrated spectrum consists entirely of a step function, and the stationary process is said to have a purely discrete spectrum (or a line spectrum). In this case the ACVS does not damp down to zero.

(c) Mixed spectrum. This case corresponds to $S_1^{(I)}(f) \geq 0$ and $S_2^{(I)}(f) \geq 0$.

- Example case (a): ARMA processes in general.
- Example case (b): sinusoid with random Phase and Amplitude.
- Example case (c): point-wise aggregation of above.

Spectral representation theorem

Complex random variables

- Note that the covariance of two complex random variables Z_1 and Z_2 is defined as

$$\mathrm{Cov}\left(Z_1, Z_2\right) = \mathbb{E}\left[\left(Z_1 - \mathbb{E}\left[Z_1\right]\right)\left(Z_2 - \mathbb{E}\left[Z_2\right]\right)^*\right].$$

  where a^* denotes the complex conjugate of $a \in \mathbb{C}$.

- The complementary covariance or relation is defined as

$$\mathrm{Rel}\{Z_1, Z_2\} = \mathbb{E}\left[(Z_1 - EZ_1)(Z_2 - EZ_2)\right].$$

Theorem 7.6 (Spectral representation theorem)

Let $\{X_t\}$ be a real-valued discrete time stationary process with mean μ. Then there exists an orthogonal increment process $\{Z(f)\}$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ such that

$$X_t = \mu + \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi ift} \mathrm{d}Z(f). \tag{7.9}$$

This equality holds in the mean–square sense. The process $\{Z(f)\}$ has properties for $f, f' \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

1. $\mathbb{E}\left[\mathrm{d}Z(f)\right] = 0$,
2. $\mathrm{Var}\left(\mathrm{d}Z(f)\right) = \mathrm{d}S^{(I)}(f)$,
3. $\mathrm{Cov}\left(\mathrm{d}Z(f), \mathrm{d}Z(f')\right) = 0$ *if* $f \neq f'$.

Bibliography

Titchmarsh, E. C. (1960). *The theory of functions*. Oxford University Press, London.