Time Series lecture 2

ARMA

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Lecture outline

1. White noise
2. Moving average models
3. Autoregressive models
4. ARMA

White noise

Example of stationary process: white noise

Definition 2.1

An example of a stationary process is a white noise, also known as a purely random process. This corresponds to a sequence $\{X_t\}$ of uncorrelated RVs such that for all $t \in \mathbb{Z}$

$$\mathbb{E}\left[X_t\right] = 0, \quad \mathrm{Var}\left(X_t\right) = \sigma^2 < \infty.$$

In this case

$$\gamma_\tau = \begin{cases} \sigma^2 & \text{if } \tau = 0, \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently

$$\rho_\tau = \begin{cases} 1 & \text{if } \tau = 0, \\ 0 & \text{otherwise.} \end{cases}$$

White noise is a building block for other time series models.

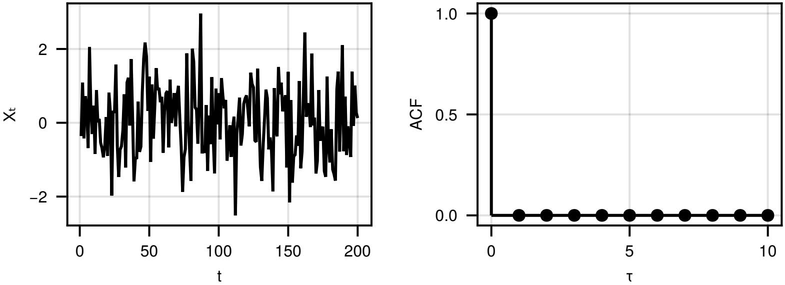

Figure: A simulated realisation of a white noise process (left) and true ACF (right)

Moving average models

Moving average processes (MA)

Definition 2.2 (MA(q))

Let $\{\varepsilon_t\}$ be a mean-zero white noise process. Then we define the q-th order moving average process, denoted MA(q), as

$$X_t = \mu - \sum_{j=0}^{q} \theta_j \varepsilon_{t-j}, \tag{2.1}$$

where θ_j are constants such that $\theta_0 = -1$ and $\theta_q \neq 0$.

- All we have done is taken a noisy process and averaged consecutive values.
- This isn't necessarily smoother than the original!

Moments of MA(q)

We can calculate the moments of this process.

- We find for the expectation that

$$\mathbb{E}\left[X_t\right] = \mu. \tag{2.2}$$

- Now for the covariance, let $\theta_j = 0$ for $j > q$, then

$$\operatorname{Cov}\left(X_{t+\tau}, X_t\right) = \begin{cases} \sigma_\varepsilon^2 \sum_{j=0}^{q} \theta_j \theta_{j+|\tau|} & \text{if } |\tau| \leq q, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Therefore, the only constraint for stationarity is that $|\theta_j| < \infty$.

Moments of MA(q): computation

$$\begin{aligned}
\operatorname{Cov}(X_{t+\tau}, X_t) &= \operatorname{Cov}\left(\mu - \sum_{k=0}^{q} \theta_k \varepsilon_{t+\tau-k}, \mu - \sum_{j=0}^{q} \theta_j \varepsilon_{t-j}\right) \\
&= \sum_{j=0}^{q} \sum_{k=0}^{q} \theta_j \theta_k \operatorname{Cov}(\varepsilon_{t+\tau-k}, \varepsilon_{t-j}) \\
&= \sigma_\varepsilon^2 \sum_{j=0}^{q} \sum_{k=0}^{q} \theta_j \theta_k \mathbb{1}_{\tau=k-j}
\end{aligned}$$

To find the terms which survive consider $0 \leq \tau \leq q$, we see that for a given j only terms $k = j + \tau$ survive. The full result follows from symmetry.

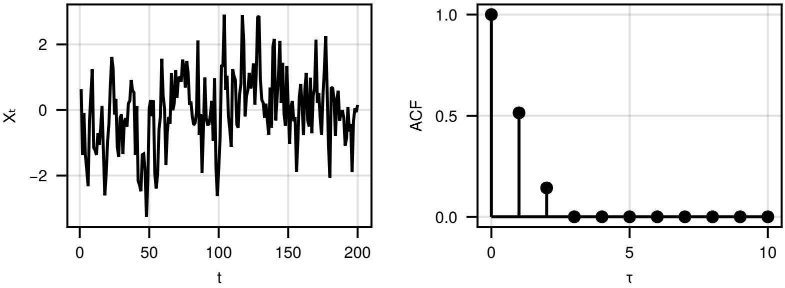

Figure: A simulated realisation of an MA(2) process (left) and true ACF (right)

An MA(1) example

Consider an MA(1) with zero mean, with ε a mean-zero white noise with variance σ_ε^2

$$X_t = \varepsilon_t - \theta\varepsilon_{t-1}.$$

The autocovariance of $\{X_t\}$ is given by

$$\gamma_\tau = \begin{cases} \sigma_\varepsilon^2\left(1+\theta^2\right) & \text{if } \tau = 0, \\ -\sigma_\varepsilon^2\theta & \text{if } |\tau| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider a different white noise process $\{\eta_t\}$ with variance $\sigma_\eta^2 = \sigma_\varepsilon^2\theta^2$ and define

$$Y_t = \eta_t - \frac{1}{\theta}\eta_{t-1}.$$

Write $\tilde{\gamma}_\tau$ for the ACVS of $\{Y_t\}$, then

$$\begin{aligned}
\tilde{\gamma}_0 &= \sigma_\varepsilon^2\theta^2\left(1 + \frac{1}{\theta}^2\right)\\
&= \sigma_\varepsilon^2\left(1+\theta^2\right)\\
\tilde{\gamma}_1 &= -\sigma_\varepsilon^2\theta^2/\theta\\
&= -\sigma_\varepsilon^2\theta
\end{aligned}$$

and $\tilde{\gamma}_{-1} = \tilde{\gamma}_1$ and the remainder being zero. So $\{\tilde{\gamma}_\tau\} = \{\gamma_\tau\}$ and we cannot identify the two processes from the autocovariance sequence.

Autoregressive models

Autoregressive process of order p, AR(p)

Definition 2.3

Let $\{\varepsilon_t\}$ be a mean-zero white noise process. Then we define the p-th order autoregressive process, denoted AR(p), as

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t, \tag{2.4}$$

where ϕ_j are constants such that $\phi_p \neq 0$.

- In contrast with the moving average process, we have constraints on $\{\phi_j\}$ to obtain a stationary process (see lecture 4).
- We also cannot give a nice closed form for the autocovariance in general.

AR(1) example

Say that we have an AR(1) process, so that

$$X_t = \phi X_{t-1} + \varepsilon_t.$$

Assume that $|\phi| < 1$. Then we may iterate and informally find that:

$$\begin{aligned} X_t &= \phi\left(\phi X_{t-2} + \varepsilon_{t-1}\right) + \varepsilon_t \\ &= \phi^2 X_{t-2} + \phi\varepsilon_{t-1} + \varepsilon_t \\ &= \phi^3 X_{t-3} + \phi^2\varepsilon_{t-2} + \phi\varepsilon_{t-1} + \varepsilon_t \\ &= \dots \\ &= \sum_{k=0}^{\infty} \phi^k \varepsilon_{t-k}. \end{aligned}$$

This statement can be made formally correct under certain conditions, where the equality can be seen to hold with probability one.

Note that the mean and covariance is given by

$$\mathbb{E}\left[X_t\right] = \sum_{k=0}^{\infty} \phi^k \cdot 0 = 0. \tag{2.5}$$

$$\begin{aligned} \operatorname{Cov}\left(X_{t+\tau}, X_t\right) &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \phi^{k+j} \operatorname{Cov}\left(\varepsilon_{j+\tau}, \varepsilon_k\right) \\ &= \sigma_\varepsilon^2 \sum_{k=0}^{\infty} \phi^{2k+|\tau|} \\ &= \frac{\sigma_\varepsilon^2}{1-\phi^2} \phi^{|\tau|}. \end{aligned} \tag{2.6}$$

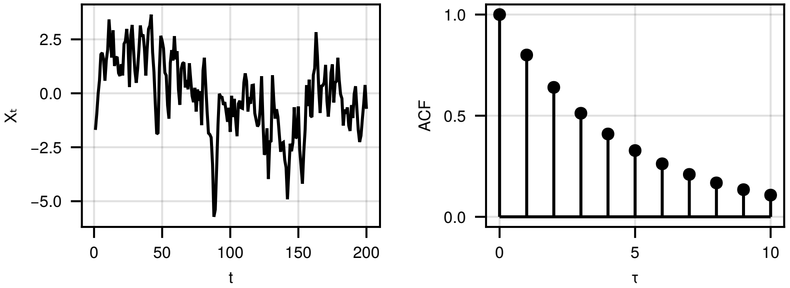

Figure: A simulated realisation of a AR(1) (left) and true ACF (right)

ARMA

Auto-regressive Moving Average Process ARMA(p, q)

Definition 2.4 (ARMA(p, q))

A time series $\{X_t\}$ is an autoregressive moving average process of order p and q, denoted ARMA(p, q), if

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} - \sum_{k=0}^{q} \theta_k \varepsilon_{t-k}$$

where $\{\varepsilon_t\}$ is a mean-zero white noise process, and ϕ_j, θ_k are the same as in the AR and MA cases respectively.

- Again we will have conditions on the AR parameters in order for this to be a valid model.
- Again general closed forms for the autocovariance aren't obtainable (though techniques to compute the autocovariance exist).

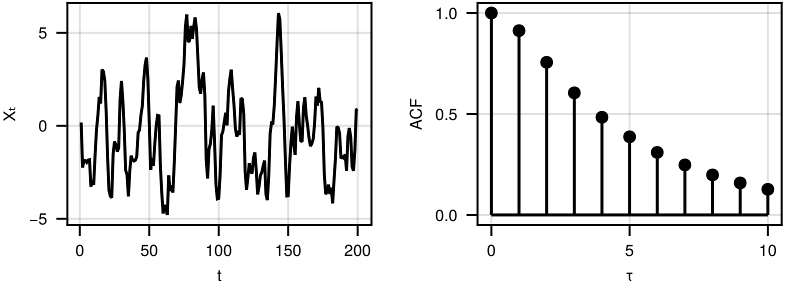

Figure: A simulated realisation of a ARMA process (left) and true ACF (right)

Causal Process

Definition 2.5 (Causal ARMA(p, q))

An ARMA(p, q) process is said to be causal (or more specifically to be a causal function of $\{\varepsilon_t\}$) if there exists a sequence of constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and

$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \quad t = 0, \pm 1, \ldots$$

- A mean-zero $MA(q)$ process is always causal.