Time Series Exercise Sheet 13

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Exercise 13.1

Assume the generation of r_t is governed by an ARCH (AutoRegressive Conditionally Heteroscedastic) model. This particular instance takes the form for β_j with $j = 0, 1, 2$ non-negative constants and ε_t a zero-mean process:

$$r_t = \sigma_t \varepsilon_t$$
$$\sigma_t^2 = \beta_0 + \beta_1 r_{t-1}^2 + \beta_2 r_{t-2}^2,$$

and ε_t is assumed to have unit variance for identifiability. Let F_t denote all the information available at time point t

(a) Calculate $\mathbb{E}[r_t]$ using the law of iterated expectation, conditioning on F_{t-1}.

(b) Assuming r_t is stationary, calculate the marginal variance of r_t, quoting any laws of probability you may need.

(c) Determine necessary conditions for r_t to be stationary in terms of β_j with $j = 0, 1, 2$.

Exercise 13.2

The generalised autoregressive conditionally heteroscedastic (GARCH) class of models is given by

$$r_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \sum_{j=1}^{m} \alpha_j r_{t-j}^2 + \sum_{j=1}^{r} \beta_j \sigma_{t-j}^2, \quad \varepsilon_t \overset{iid}{\sim} \mathcal{N}(0, 1) \tag{1}$$

and labelled GARCH(m, r). To avoid negative variances, we take $\alpha_j, \beta_j \geq 0$.

A GARCH$(m, 0)$ is an ARCH(m). Prove that a GARCH model has zero mean, and satisfies: $\text{Corr}(r_{t+\tau}, r_t) = 0$.

Remember that F_τ denote the entire history of the process $\{r_t\}$ up to time τ.

Exercise 13.3

Show that for an ARCH(1) with parameters $0 < \alpha_0,\ 0 < \alpha_1 < 1/3$ we have $\text{Corr}\left(r_{+\tau}^2, r_t^2\right) = \alpha_1^{|\tau|}$