Time Series Exercise Sheet 9

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Exercise 9.1

For the process $\mathbf{Y}_t = \left(Y_t^{(1)},\ Y_t^{(2)}\right)^T$ generated by

$$Y_t^{(j)} = X_t^{(j)} + \varepsilon_t, \quad j = 1, 2$$

with $X^{(1)}$, $X^{(2)}$ and ε independent and where ε is white noise. Write

$$\begin{aligned}
\sigma_j^2 &= \operatorname{Var}\left(X_t^{(j)}\right) \\
\sigma_\epsilon^2 &= \operatorname{Var}\left(\varepsilon_t\right) \\
\tilde{\gamma}_\tau^{(1,2)} &= \operatorname{Cov}\left(X_{t+\tau}^{(1)}, X_t^{(2)}\right)
\end{aligned}$$

Use $\gamma_\tau^{(1,2)}$ to denote the cross-covariance sequence between $\left\{Y_t^{(1)}\right\}$ and $\left\{Y_t^{(2)}\right\}$. Calculate the correlation between $\left\{Y_t^{(1)}\right\}$ and $\left\{Y_t^{(2)}\right\}$ at all lags.

Solution 9.1

Since ϵ_t is white noise, and noting that $X_{j,t}$ and ϵ_t are independent, we have

$$\begin{aligned}
\gamma_\tau^{(1,2)} &= \operatorname{Cov}\left(Y_{t+\tau}^{(1)}, Y_{t+\tau}^{(2)}\right) \\
&= \operatorname{Cov}\left(X_{t+\tau}^{(1)}, X_t^{(2)}\right) + \operatorname{Cov}\left(X_{t+\tau}^{(1)}, \epsilon_t\right) + \operatorname{Cov}\left(\epsilon_{t+\tau}, X_t^{(2)}\right) + \operatorname{Cov}\left(\epsilon_{t+\tau}, \epsilon_t\right) \\
&= \operatorname{Cov}\left(X_{t+\tau}^{(1)}, X_t^{(2)}\right) + \operatorname{Cov}\left(\epsilon_{t+\tau}, \epsilon_t\right) \\
&= \begin{cases} \tilde{\gamma}_0^{(1,2)} + \sigma_\epsilon^2 & \text{if } \tau = 0, \\ \tilde{\gamma}_\tau^{(1,2)} & \text{if } \tau \neq 0, \end{cases}
\end{aligned}$$

and marginally

$$\begin{aligned}
\gamma_0^{(1,1)} &= \operatorname{Cov}\left(Y_t^{(1)}, Y_t^{(1)}\right) \\
&= \operatorname{Cov}\left(X_t^{(1)}, X_t^{(1)}\right) + \operatorname{Cov}\left(\epsilon_t, \epsilon_t\right) \\
&= \sigma_1^2 + \sigma_\epsilon^2,
\end{aligned}$$

finally we have

$$\gamma_0^{(2,2)} = \sigma_2^2 + \sigma_\epsilon^2.$$

Therefore,

$$\rho_\tau^{(1,2)} = \begin{cases} \dfrac{\tilde{\gamma}_0^{(1,2)} + \sigma_\epsilon^2}{\sqrt{\sigma_1^2 + \sigma_\epsilon^2}\sqrt{\sigma_2^2 + \sigma_\epsilon^2}} & \text{if } \tau = 0, \\ \dfrac{\tilde{\gamma}_\tau^{(1,2)}}{\sqrt{\sigma_1^2 + \sigma_\epsilon^2}\sqrt{\sigma_2^2 + \sigma_\epsilon^2}} & \text{otherwise.} \end{cases}$$

Exercise 9.2

For the delay process, with $\alpha > 0$ and $\nu \in \mathbb{Z}$

$$X_t^{(1)} = \alpha X_{t-\nu}^{(2)} + \varepsilon_t, \quad t \in \mathbb{Z}$$

where $\{\varepsilon_t\}$ is white noise and $\left\{X_t^{(2)}\right\}$ is stationary and uncorrelated with $\{\varepsilon_t\}$ at all lags

(a) Calculate the correlation between $\left\{X_t^{(1)}\right\}$ and $\left\{X_t^{(2)}\right\}$ at all lags.

(b) Calculate the cross spectrum, and determine the amplitude (absolute value) and phase (argument) of the cross-spectrum.

Solution 9.2

(a) Again noting that $X_t^{(j)}$ and ϵ_t are independent, we have

$$\begin{aligned}
\gamma_\tau^{(1,2)} &= \mathrm{Cov}\left(X_{t+\tau}^{(1)}, X_t^{(2)}\right) \\
&= \alpha\,\mathrm{Cov}\left(X_{t+\tau-\nu}^{(2)}, X_t^{(2)}\right) + \mathrm{Cov}\left(\epsilon_{t+\tau}, X_t^{(2)}\right) \\
&= \alpha\,\mathrm{Cov}\left(X_{t+\tau-\nu}^{(2)}, X_t^{(2)}\right) \\
&= \alpha \gamma_{\tau-\nu}^{(2,2)},
\end{aligned}$$

additionally,

$$\begin{aligned}
\gamma_0^{(1,1)} &= \alpha^2\sigma_2^2 + \sigma_\epsilon^2, \\
\gamma_0^{(2,2)} &= \sigma_2^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho_\tau^{(1,2)} &= \frac{\alpha\gamma_{\tau-\nu}^{(2,2)}}{\sqrt{\alpha^2\sigma_2^2+\sigma_\epsilon^2}\sqrt{\sigma_2^2}} \\
&= \frac{\rho_{\tau-\nu}^{(2,2)}}{\sqrt{1+\sigma_\epsilon^2/(\alpha\sigma_2)^2}}
\end{aligned}$$

(b) We have

$$\begin{aligned}
S_{1,2}(f) &= \sum_{\tau=-\infty}^{\infty} \gamma_\tau^{(1,2)} e^{-2\pi i f\tau} \\
&= \alpha \sum_{\tau=-\infty}^{\infty} \gamma_{\tau-\nu}^{(2,2)} e^{-2\pi i f\tau} \\
&= \alpha e^{-2\pi i f\nu} \sum_{\tau=-\infty}^{\infty} \gamma_\tau^{(2,2)} e^{-2\pi i f\tau} \\
&= \alpha S_{2,2}(f)\, e^{-2\pi i f\nu}
\end{aligned}$$

showing that the amplitude is the spectral density of $X_t^{(2)}$ scaled by α and the phase is $f \mapsto -2\pi f\nu$.

Exercise 9.3

For $\mathbf{Y}_t = \begin{pmatrix} Y_t^{(1)} \\ Y_t^{(2)} \end{pmatrix} \sim N(\mathbf{0}, \mathbf{\Sigma})$ independently in t with $\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$, determine the expectation and variance of $\widehat{\gamma}_\tau^{(1,2)}$ in

$$\widehat{\mathbf{\Gamma}}_\tau = \begin{bmatrix} \widehat{\gamma}_\tau^{(1,1)} & \widehat{\gamma}_\tau^{(1,2)} \\ \widehat{\gamma}_\tau^{(2,1)} & \widehat{\gamma}_\tau^{(2,2)} \end{bmatrix} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} \mathbf{Y}_{t+\tau}\mathbf{Y}_t^T$$

Solution 9.3

$$\begin{aligned}
\mathbb{E}\left[\widehat{\gamma}_\tau^{(1,2)}\right] &= \mathbb{E}\left[\frac{1}{N}\sum_{t=1}^{N-|\tau|} Y_{t+\tau}^{(1)} Y_t^{(2)}\right] \\
&= \frac{1}{N}\sum_{t=1}^{N-|\tau|} \mathbb{E}\left[Y_{t+\tau}^{(1)} Y_t^{(2)}\right] \\
&= \begin{cases} \sigma_{12} & \tau = 0 \\ 0 & \tau \neq 0. \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(\widehat{\gamma}_\tau^{(1,2)}) &= \mathrm{Var}\left(\frac{1}{N}\sum_{t=1}^{N-|\tau|} Y_{t+\tau}^{(1)} Y_t^{(2)}\right) \\
&= \mathbb{E}\left[\left(\frac{1}{N}\sum_{t=1}^{N-|\tau|} Y_{t+\tau}^{(1)} Y_t^{(2)}\right)^2\right] - \mathbb{E}\left[\widehat{\gamma}_\tau^{(1,2)}\right]^2 \\
&= \frac{1}{N^2}\left\{\sum_{t=1}^{N-|\tau|} \mathbb{E}\left[\left(Y_{t+\tau}^{(1)}\right)^2 \left(Y_t^{(2)}\right)^2\right] + \sum_{t\neq s}^{N-|\tau|} \mathbb{E}\left[Y_{t+\tau}^{(1)} Y_t^{(2)} Y_{s+\tau}^{(1)} Y_s^{(2)}\right]\right\} - \{\mathbb{E}(\widehat{\gamma}_\tau^{(1,2)})\}^2.
\end{aligned}$$

Note that $\mathbb{E}\left[\left(Y_t^{(1)}\right)^2 \left(Y_t^{(2)}\right)^2\right] = \sigma_{11}\sigma_{22} + 2\sigma_{12}^2$ since both $Y_t^{(1)}$ and $Y_t^{(2)}$ are Gaussian. Therefore, if $\tau = 0$,

$$\begin{aligned}
\mathrm{Var}\left(\widehat{\gamma}_0^{(1,2)}\right) &= \frac{1}{N^2}\left\{\sum_{t=1}^{N} \mathbb{E}\left[\left(Y_t^{(1)}\right)^2 \left(Y_t^{(2)}\right)^2\right] + \sum_{t\neq s}^{N} \mathbb{E}\left[Y_t^{(1)} Y_t^{(2)} Y_s^{(1)} Y_s^{(2)}\right]\right\} - \sigma_{12}^2 \\
&= \frac{1}{N^2}\left\{\sum_{t=1}^{N} \mathbb{E}\left[\left(Y_t^{(1)}\right)^2 \left(Y_t^{(2)}\right)^2\right] + \sum_{t\neq s}^{N} \mathbb{E}\left[Y_t^{(1)} Y_t^{(2)} Y_s^{(1)} Y_s^{(2)}\right]\right\} - \sigma_{12}^2 \\
&= \frac{1}{N}(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) + \frac{N-1}{N}\sigma_{12}^2 - \sigma_{12}^2 \\
&= \frac{1}{N}(\sigma_{11}\sigma_{22} + \sigma_{12}^2).
\end{aligned}$$

If $\tau \neq 0$,

$$\begin{aligned}
\mathrm{Var}(\widehat{\gamma}_\tau^{(1,2)}) &= \frac{1}{N^2}\left\{\sum_{t=1}^{N-|\tau|} \mathbb{E}\left[\left(Y_{t+\tau}^{(1)}\right)^2 \left(Y_t^{(2)}\right)^2\right] + \sum_{t\neq s}^{N-|\tau|} \mathbb{E}\left[Y_{t+\tau}^{(1)} Y_t^{(2)} Y_{s+\tau}^{(2)} Y_s^{(2)}\right]\right\} \\
&= \frac{1}{N^2}\left\{(N-|\tau|)\sigma_{11}\sigma_{22} + 0\right\} \\
&= \frac{N-|\tau|}{N^2}\sigma_{11}\sigma_{12}.
\end{aligned}$$

In summary,

$$\begin{aligned}
\mathbb{E}\left[\widehat{\gamma}_\tau^{(1,2)}\right] &= \sigma_{12}\delta_\tau, \\
\mathrm{Var}(\widehat{\gamma}_\tau^{(1,2)}) &= \frac{N-|\tau|}{N^2}(\sigma_{11}\sigma_{22} + \sigma_{12}^2\delta_\tau),
\end{aligned}$$

where

$$\delta_\tau = \begin{cases} 1 & \tau = 0, \\ 0 & \tau \neq 0. \end{cases}$$