Time Series Exercise Sheet 8

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Exercise 8.1

Say that we observe X_t for $t \in T$ where $T = \{1, \ldots, n\}$. Recall that the periodogram is

$$\widehat{S}^{(p)}(f) = \sum_{\tau \in \mathbb{Z}} \hat{\gamma}_\tau e^{-2\pi i f \tau}. \tag{1}$$

Show that

$$\widehat{S}^{(p)}(f) = |J(f)|^2, \tag{2}$$

where

$$J(f) = \sqrt{\frac{1}{n}} \sum_{t \in T} \left(X_t - \bar{X}\right) e^{-2\pi i t f}. \tag{3}$$

Solution 8.1

Recall that

$$\hat{\gamma}_\tau = \begin{cases} \frac{1}{n} \sum_{t=1}^{n-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X}) & \text{if } |\tau| \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

For notational convenience, define $Y_t = X_t - \bar{X}$. Now we have

$$\begin{aligned} \widehat{S}^{(p)}(f) &= \sum_{\tau=-n+1}^{n-1} \frac{1}{n} \sum_{t=1}^{n-|\tau|} Y_t Y_{t+|\tau|} e^{-2\pi i \tau f} \\ &= \frac{1}{n} \sum_{s=1}^{n} \sum_{t=1}^{n} Y_t Y_s e^{-2\pi i s f} e^{2\pi i t f} \\ &= |J(f)|^2. \end{aligned}$$

Exercise 8.2

Recall that $u = g * h$ means

$$u_t = \sum_{s \in \mathbb{Z}} g_{t-s} h_s \tag{4}$$

for all $t \in \mathbb{Z}$ and $V = G * H$ means

$$V(f) = \int_{-1/2}^{1/2} G(f - f') H(f') \mathrm{d}f' \tag{5}$$

Assume that $g, h \in \ell^1$, prove the following:

(a) $h \cdot g \in \ell^1$ and $h * g \in \ell^1$,

(b) $H \in L^1$,

(c) If $v_t = h_t \cdot g_t$, then the Fourier transform of v is given by $V = H * G$,

(d) If $u = h * g$ then the Fourier transform of u is given by $U = H \cdot G$.

(Note, here L^1 refers to functions with domain $[-1/2, 1/2]$.)

Solution 8.2

(a) Firstly, consider $v = h \cdot g$, then we have

$$\begin{aligned} \|v\|_1 &= \sum_{t\in\mathbb{Z}} |v_t| \\ &= \sum_{t\in\mathbb{Z}} |h_t g_t| \\ &\leq \sum_{t\in\mathbb{Z}} |h_t| \sum_{s\in\mathbb{Z}} |g_s| \\ &= \|h\|_1 \|g\|_1 \\ &< \infty. \end{aligned}$$

Secondly, consider $u = h * g$

$$\begin{aligned} \|u\|_1 &= \sum_{t\in\mathbb{Z}} |u_t| \\ &= \sum_{t\in\mathbb{Z}} \left| \sum_{s\in\mathbb{Z}} g_{t-s} h_s \right| \\ &\leq \sum_{t\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} |g_{t-s} h_s| \\ &= \sum_{s\in\mathbb{Z}} |h_s| \sum_{x\in\mathbb{Z}} |g_x| \\ &= \|h\|_1 \|g\|_1 \\ &< \infty, \end{aligned}$$

where the interchange of summation is justified because $h, g \in \ell^1$.

(b) We have for any $f \in [-1/2, 1/2]$

$$\begin{aligned} |H(f)| &= \left| \sum_{t\in\mathbb{Z}} h_t e^{-2\pi i t f} \right| \\ &\leq \sum_{t\in\mathbb{Z}} |h_t| \\ &= \|h_t\|_1 \\ &< \infty. \end{aligned}$$

Therefore,

$$\int_{-1/2}^{1/2} |G(f)| \, \mathrm{d}f \leq \|g_t\|_1 < \infty$$

as required.

(c) Letting $v = h \cdot g$,

$$\begin{aligned}
V(f) &= \sum_{t\in\mathbb{Z}} v_t e^{-2\pi i t f} \\
&= \sum_{t\in\mathbb{Z}} h_t g_t e^{-2\pi i t f} \\
&= \sum_{t\in\mathbb{Z}} \int_{-1/2}^{1/2} H(f') e^{2\pi i t f'} \mathrm{d}f' g_t e^{-2\pi i t f} \\
&= \int_{-1/2}^{1/2} H(f) \sum_{t\in\mathbb{Z}} g_t e^{-2\pi i t (f-f')} \mathrm{d}f' \\
&= \int_{-1/2}^{1/2} H(f') G(f-f') \mathrm{d}f'
\end{aligned}$$

where we use the Fourier inverse since $h \in \ell^1$, and interchanging of the sum and integral is justified because $H \in L^1$ and $g \in \ell^1$.

(d) Let $u = g * h$, then its Fourier transform is

$$\begin{aligned}
U(f) &= \sum_{t\in\mathbb{Z}} u_t e^{-2\pi i t f} \\
&= \sum_{t\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} g_{t-s} h_s e^{-2\pi i (t-s) f} e^{-2\pi i s f} \\
&= \sum_{s\in\mathbb{Z}} h_s e^{-2\pi i s f} \sum_{t\in\mathbb{Z}} g_{t-s} e^{-2\pi i (t-s) f} \\
&= \sum_{s\in\mathbb{Z}} h_s e^{-2\pi i s f} \sum_{x\in\mathbb{Z}} g_x e^{-2\pi i x f} \\
&= H(f) G(f)
\end{aligned}$$

where interchanging terms in the summation is justified because $h, g \in \ell^1$.

Exercise 8.3

Replacing $\bar{X}$ with μ, show that

$$\mathbb{E}\left[\widehat{S}_h^{(p)}(f)\right] = \int_{-1/2}^{1/2} S(f') \left|H(f-f')\right|^2 \mathrm{d}f' \tag{6}$$

where H is the discrete Fourier transform of h.

Solution 8.3

Again, relabel with $Y_t = X_t - \mu$ and $g_t = h_t$ for all $t \in \{1, \ldots, n\}$. Now we have

$$\begin{aligned}
\widehat{S}_h^{(p)}(f) &= \left| \sum_{t=1}^{n} g_t Y_t e^{-2\pi i f t} \right|^2 \\
&= \sum_{t=1}^{n} \sum_{s=1}^{n} g_t g_s Y_t Y_s e^{-2\pi i f (t-s)} \\
&= \sum_{\tau=-n+1}^{n-1} \sum_{t=1}^{n-|\tau|} g_t g_{t+|\tau|} Y_t Y_{t+\tau} e^{-2\pi i f \tau}.
\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}\left[\widehat{S}_h^{(p)}(f)\right] &= \sum_{\tau=-n+1}^{n-1}\sum_{t=1}^{n-|\tau|} h_t h_{t+|\tau|}\gamma_\tau e^{-2\pi i f\tau} \\ &= \sum_{\tau\in\mathbb{Z}} w_\tau \gamma_\tau e^{-2\pi i f\tau}\end{aligned}$$

where if we define $h_t = 0$ for $t \in \mathbb{Z}\setminus T$

$$w_\tau = \sum_{t\in\mathbb{Z}} h_t h_{t+\tau}.$$

By the convolution theorem, we have that

$$\mathbb{E}\left[\widehat{S}_h^{(p)}(f)\right] = \int_{-1/2}^{1/2} S(f')\, W(f-f')\mathrm{d}f'$$

where W is the discrete Fourier transform of w. Now setting for all $t \in \mathbb{Z}$, $\tilde{h}_t = h_{-t}$ and $\bar{h}_t = h_t$, we see that $w = \tilde{h} * \bar{h}$, and so by application of the convolution theorem again

$$\begin{aligned}W(f) &= \tilde{H}(f)\bar{H}(f) \\ &= |H(f)|^2.\end{aligned}$$

This gives the desired result.

Exercise 8.4

Consider the case that $\{X_t\}$ is a white noise process with variance σ^2. Prove that $\widehat{S}_h^{(p)}(f)$ is unbiased for all n if $\|h\|_2^2 = 1$.

Solution 8.4

Recall that the spectral density function of a white noise if $S(f) = \sigma^2$. Thus have for any $f \in \mathbb{R}$

$$\begin{aligned}\mathbb{E}\left[\widehat{S}_h^{(p)}(f)\right] &= \int_{-1/2}^{1/2} S(f')\,|H(f-f')|^2\,\mathrm{d}f' \\ &= \int_{-1/2}^{1/2} S(f-f')\,|H(f')|^2\,\mathrm{d}f' && \text{(relabelling and periodicity)} \\ &= \sigma^2 \int_{-1/2}^{1/2} |H(f')|^2\,\mathrm{d}f' && \text{(because white noise)} \\ &= \sigma^2 \sum_{t\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} h_t h_s \int_{-1/2}^{1/2} e^{-2\pi i(t-s)f'}\mathrm{d}f' && \text{(definition of discrete Fourier transform)} \\ &= \sigma^2 \cdot \|h\|_2^2\end{aligned}$$

where the last line follows because if $\tau \in \mathbb{Z}$

$$\int_{-1/2}^{1/2} e^{-2\pi i\tau f}\mathrm{d}f = \begin{cases} 1 & \text{if } \tau = 0, \\ 0 & \text{if } \tau \neq 0. \end{cases}$$

Therefore, if $\|h\|_2^2 = 1$ then

$$\begin{aligned}\mathbb{E}\left[\widehat{S}_h^{(p)}(f)\right] &= \sigma^2 \\ &= S(f),\end{aligned}$$

as required.