# Time Series Exercise Sheet 7

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April 3, 2025

## Exercise 7.1

Let $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ be a white noise process with variance $\sigma_\varepsilon^2$. Determine the spectral density function.

## Solution 7.1

If $\varepsilon_t$ is a white noise, then the process is uncorrelated so $\gamma_\tau = \sigma_\varepsilon^2$ if $\tau = 0$, and 0 otherwise. Therefore, the spectral density of $\{\varepsilon_t\}$ is for all $f$

$$S(f) = \sigma_\varepsilon^2.$$

## Exercise 7.2

Consider the sequence

$$\gamma_\tau = \begin{cases} 1 & \text{if } |\tau| \le K \\ 0 & \text{if } |\tau| > K \end{cases}$$

where $K$ is a positive integer. Is $\gamma_\tau$ an ACVS for some discrete parameter stationary process $\{Y_t\}$? What would be the SDF?

## Solution 7.2

Note first that $\sum_{\tau\in\mathbb{Z}} |\gamma_\tau| < \infty$ and $\gamma_\tau = \gamma_{-\tau}$, and that we can define the function $S(\cdot)$ such that for all $f \in [-1/2, 1/2]$

$$\begin{aligned} S(f) &= \sum_{\tau=-\infty}^{\infty} \gamma_\tau \exp(-2\pi i\tau f) \\ &= \sum_{\tau=-K}^{K} \exp(-2\pi i\tau f) \\ &= 1 + 2\sum_{\tau=1}^{K} \cos(2\pi\tau f) \end{aligned}$$

However, this does not satisfy $S(f) \ge 0$ for all $f \in [-1/2, 1/2]$, so $\gamma_\tau$ cannot be an ACVS for a stationary time series. Indeed, if $K$ is odd, i.e. $K = 2q + 1$ for $q \in \mathbb{Z}$, we have $S(1/2) < 0$.

## Exercise 7.3

Determine the auto-covariance function of $\{X_t\}_{t\in\mathbb{Z}}$ if it has spectrum

$$S(f) = \sigma^2 \frac{(1 - 2|f|)}{\pi} \mathbb{1}(|f| \le 1/2)$$

## Solution 7.3

We compute for $|\tau| > 0$:

$$\begin{aligned}
\gamma_\tau &= \int_{-1/2}^{1/2} e^{2\pi i f\tau} S(f)\,\mathrm{d}f \\
&= \frac{\sigma^2}{\pi}\int_{-1/2}^{1/2} e^{2\pi i f\tau}(1-2|f|)\mathrm{d}f \\
&= \frac{\sigma^2}{\pi}\int_{-1/2}^{1/2} (\cos(2\pi f\tau) + i\sin(2\pi f\tau))(1-2|f|)\mathrm{d}f \\
&= \frac{\sigma^2}{\pi}\int_{-1/2}^{1/2} \cos(2\pi f\tau)(1-2|f|)\mathrm{d}f && (\sin \text{ is odd}) \\
&= \frac{2\sigma^2}{\pi}\int_{0}^{1/2} \cos(2\pi f\tau)(1-2f)\mathrm{d}f && (\text{Integrand is even}) \\
&= \frac{2\sigma^2}{\pi}\Big\{ \underbrace{\int_0^{1/2}\cos(2\pi f\tau)\mathrm{d}f}_{I_1} - 2\underbrace{\int_0^{1/2}\cos(2\pi f\tau) f\,\mathrm{d}f}_{I_2} \Big\}.
\end{aligned}$$

Then $I_1 = \frac{\sin(\pi\tau)}{2\pi\tau}$, and after computing the second integral $I_2 = \frac{\sin(\pi\tau)}{4\pi\tau} + \frac{\cos(\pi\tau)-1}{4\pi^2\tau^2}$. Summing up both we end up with

$$\gamma_\tau = \frac{\sigma^2}{\pi^3}\{\frac{1-\cos(\pi\tau)}{\tau^2}\}.$$

For $\tau = 0$, we compute $\gamma_0 = \frac{\sigma^2}{2\pi}$.

## Exercise 7.4

What is the spectral density function of an MA($q$) process?

## Solution 7.4

Recall that for an MA($q$) we have autocovariance function given by

$$\gamma_\tau = \begin{cases} \sigma_\varepsilon^2 \sum_{j=0}^{q} \theta_j\theta_{j+|\tau|} & \text{if } |\tau| \le q, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta_j = 0$ for $j > q$. In fact, looking at how this was computed we have the more useful equation:

$$\gamma_\tau = \sigma_\varepsilon^2 \sum_{j=0}^{q}\sum_{k=0}^{q} \theta_j\theta_k \mathbb{1}_{\tau=k-j}.$$

Clearly $\gamma \in \ell^1$, so the spectral density function exists and can be computed by taking the Fourier transform of the ACVS. Thus the spectral density function is

$$\begin{aligned}
S(f) &= \sum_{\tau\in\mathbb{Z}} \gamma_\tau e^{-2\pi i\tau f} \\
&= \sum_{\tau=-\infty}^{\infty} \sigma_\varepsilon^2 \sum_{j=0}^{q}\sum_{k=0}^{q} \theta_j\theta_k \mathbb{1}_{\tau=k-j} e^{-2\pi i\tau f} \\
&= \sigma_\varepsilon^2 \sum_{j=0}^{q}\sum_{k=0}^{q} \theta_j\theta_k e^{-2\pi i(k-j)f} \\
&= \sigma_\varepsilon^2 \left| \sum_{j=0}^{q} \theta_j e^{-2\pi i j f} \right|^2.
\end{aligned}$$

## Exercise 7.5

Consider a harmonic process with a time varying amplitude

$$X_t = \epsilon_t \cos(\nu t + \Theta)$$

where we assume that $\nu$ is fixed, $\Theta$ is a random phase, and $\epsilon_t$ is not constrained to be positive, but that its a mean-zero white noise process with variance $\sigma^2$. Moreover, assume that $\epsilon_t$ is independent of $\Theta$. The harmonic process in the lecture stipulated that $\Theta$ had to be uniformly distributed on $[-\pi, \pi]$. Suppose (instead) that $\Theta$ has a pdf given by

$$f_\Theta(\theta) = \frac{1}{2\pi}(1 + \cos(\theta)), \quad |\theta| \le \pi$$

Determine the first two moments of $X_t$. Are they still independent of $t$?

## Solution 7.5

Using the independence between $\epsilon_t$ and $\Theta$, we have

$$\mathbb{E}[X_t] = \mathbb{E}[\epsilon_t]\,\mathbb{E}[\cos(\nu t + \Theta)] = 0 \cdot \mathbb{E}[\cos(\nu t + \Theta)] = 0$$

which is constant in $t$, as the second expectation is finite since cos is bounded above by unity. Again we have for all $t, \tau \in T$

$$\mathbb{E}[X_t X_{t+\tau}] = \mathbb{E}[\epsilon_t \epsilon_{t+\tau}]\,\mathbb{E}[\cos(\nu t + \Theta)\cos(\nu(t+\tau) + \Theta)]$$

Since $\epsilon_t$ is uncorrelated, we have $\mathbb{E}[\epsilon_t \epsilon_{t+\tau}] = \sigma^2$ if $\tau = 0$, and 0 if $\tau \neq 0$. Using the following identity,

$$\cos(A)\cos(B) = \frac{1}{2}\{\cos(A - B) + \cos(A + B)\}$$

we have

$$\mathbb{E}[\cos(\nu t + \Theta)\cos\{\nu(t+\tau) + \Theta\}] = \frac{1}{2}[\cos(\nu\tau) + \mathrm{E}[\cos\{\nu(2t+\tau) + 2\Theta\}]]$$

and

$$\begin{aligned}
\mathbb{E}[\cos\{\nu(2t+\tau) + 2\Theta\}] &= \int_{-\pi}^{\pi} \cos\{\nu(2t+\tau) + 2\theta\} f_\Theta(\theta)\,\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\{\nu(2t+\tau) + 2\theta\}\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\{\nu(2t+\tau) + 2\theta\}\cos(\theta)\,\mathrm{d}\theta \\
&= \frac{1}{4\pi}\sin\{\nu(2t+\tau) + 2\theta\}\Big|_{-\pi}^{\pi} + \frac{1}{4\pi}\int_{-\pi}^{\pi} \cos\{\nu(2t+\tau) + \theta\}\,\mathrm{d}\theta \\
&\qquad + \frac{1}{4\pi}\int_{-\pi}^{\pi} \cos\{\nu(2t+\tau) + 3\theta\}\,\mathrm{d}\theta \\
&= 0 + \frac{1}{4\pi}\sin\{\nu(2t+\tau) + \theta\}\Big|_{-\pi}^{\pi} + \frac{1}{12\pi}\sin\{\nu(2t+\tau) + 3\theta\}\Big|_{-\pi}^{\pi} \\
&= 0 + 0 + 0 \\
&= 0.
\end{aligned}$$

We finally get $\mathbb{E}[X_t X_{t+\tau}] = \mathbb{E}[\epsilon_t \epsilon_{t+\tau}]\cos(\nu\tau)/2 = \sigma^2/2$ if $\tau = 0$ and 0 otherwise, and therefore $X_t$ is stationary.