Time Series Exercise Sheet 6

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Exercise 6.1

Let $g \in L^1$, have Fourier transform denoted by $\mathscr{G}$. Prove the following relations (where $\mathscr{H}$ is taken to be the Fourier transform of h).

1. If $h(t) = g(t)^*$ for all $t \in \mathbb{R}$ then
$$\mathscr{H}(f) = \mathscr{G}(-f)^*$$
for all $f \in \mathbb{R}$.
2. If $h(t) = g(\alpha t)$ for all $t \in \mathbb{R}$ then
$$\mathscr{H}(f) = \mathscr{G}(f/\alpha)/\left|\alpha\right|$$
for all $f \in \mathbb{R}$.
3. If $h(t) = g(t+\tau)$ for all $t \in \mathbb{R}$ then
$$\mathscr{H}(f) = \mathscr{G}(f)e^{2\pi i f\tau}$$
for all $f \in \mathbb{R}$.
4. Let $p \in L^1$ with Fourier transform $\mathscr{P}$. If $h(t) = \alpha g(t) + p(t)$ for all $t \in \mathbb{R}$ then
$$\mathscr{H}(f) = \alpha\mathscr{G}(f) + \mathscr{P}(f)$$
for all $f \in \mathbb{R}$.

Solution 6.1

1. We have $h(t) = g(t)^*$
$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)e^{-2\pi i t f}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} g(t)^* e^{-2\pi i t f}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} g(t)e^{2\pi i t f}\mathrm{d}t^* \\ &= G(-f)^*. \end{aligned}$$
2. We have $h(t) = g(\alpha t)$, then
$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)e^{-2\pi i t f}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} g(\alpha t)e^{-2\pi i t f}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} g(t')e^{-i\frac{f}{\alpha}t'}\frac{1}{|\alpha|}dt' \\ &= G(f/\alpha)/\left|\alpha\right| \end{aligned}$$

3. We have $h(t) = g(t+\tau)$, we have

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} \mathrm{d}t \\ &= \int_{-\infty}^{\infty} g(t+\tau) e^{-2\pi i f t} \mathrm{d}t \\ &= \int_{-\infty}^{\infty} g(t+\tau) e^{-2\pi i f (t+\tau)} \mathrm{d}t e^{2\pi i f \tau} \\ &= G(f) e^{2\pi i f \tau} \end{aligned}$$

4. This follows from linearity of integration.

Exercise 6.2

Prove the continuous time version of the convolution theorem, i.e. let $h, g \in L^1$, and let $u = g * h$, then

$$\mathscr{U} = \mathscr{G} \cdot \mathscr{H}.$$

Solution 6.2

We have for any $f \in \mathbb{R}$

$$\begin{aligned} U(f) &= \int_{-\infty}^{\infty} u(x) e^{-2\pi i x f} \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x-y) h(y) \mathrm{d}y \; e^{-2\pi i x f} \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x-y) e^{-2\pi i x f} \mathrm{d}x \; h(y) \mathrm{d}y \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s) e^{-2\pi i s f} \mathrm{d}x \; h(y) e^{-2\pi i y f} \mathrm{d}y \\ &= G(f) H(f). \end{aligned}$$

The interchange of integrals is justified because

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x-y) h(y)| \, \mathrm{d}x \mathrm{d}y &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(s) h(y)| \, \mathrm{d}s \mathrm{d}y \\ &= \int_{-\infty}^{\infty} |g(s)| \, \mathrm{d}s \int_{-\infty}^{\infty} |h(y)| \, \mathrm{d}y \\ &< \infty \end{aligned}$$

by assumption, and so we may apply Fubini's theorem.

Exercise 6.3

Prove the following theorem. Consider a continuous function $g \in L^1$. Then writing $\mathscr{G}$ for its Fourier transform and G for the Fourier transform of the sequence $\{g(t)\}_{t \in \Delta\mathbb{Z}}$ and assuming that $\mathscr{G} \in L^1$ we have

$$G(f) = \sum_{k \in \mathbb{Z}} \mathscr{G}(f + k/\Delta). \tag{1}$$

Solution 6.3

Firstly, recall that we have the Fourier inverse results so that for any $t \in \Delta\mathbb{Z}$

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} \mathscr{G}(f) e^{2\pi i f t} \mathrm{d}f, \\ g(t) &= \int_{-1/2\Delta}^{1/2\Delta} G(f) e^{2\pi i f t} \mathrm{d}f. \end{aligned}$$

Now notice that

$$\begin{aligned}
g(t) &= \int_{-\infty}^{\infty} \mathscr{G}(f)e^{2\pi i f t}\mathrm{d}f \\
&= \sum_{k\in\mathbb{Z}} \int_{k/\Delta-1/2\Delta}^{k/\Delta+1/2\Delta} \mathscr{G}(f)e^{2\pi i f t}\mathrm{d}f \\
&= \sum_{k\in\mathbb{Z}} \int_{-1/2\Delta}^{1/2\Delta} \mathscr{G}(f+k/\Delta)e^{2\pi i f t}\mathrm{d}f e^{2\pi i t k/\Delta} \\
&= \sum_{k\in\mathbb{Z}} \int_{-1/2\Delta}^{1/2\Delta} \mathscr{G}(f+k/\Delta)e^{2\pi i f t}\mathrm{d}f
\end{aligned}$$

because $t = z\Delta$ for some $\Delta \in \mathbb{Z}$ and $e^{2\pi i x} = 1$ for any $x \in \mathbb{Z}$. Now we may swap the order of summation and integration since

$$\sum_{k\in\mathbb{Z}} \int_{-1/2\Delta}^{1/2\Delta} |\mathscr{G}(f+k/\Delta)|\,\mathrm{d}f < \infty$$

by assumption. Therefore we have

$$\int_{-1/2\Delta}^{1/2\Delta} \sum_{k\in\mathbb{Z}} \mathscr{G}(f+k/\Delta)e^{2\pi i f t}\mathrm{d}f = g(t) = \int_{-1/2\Delta}^{1/2\Delta} G(f)e^{2\pi i f t}\mathrm{d}f.$$

It therefore follows that

$$\sum_{k\in\mathbb{Z}} \mathscr{G}(f+k/\Delta) = G(f)$$

for almost every $f \in [-1/2\Delta, 1/2\Delta]$. Finally, we have that G is continuous because g is continuous and in L^1 and therefore the sequence $\{g(t)\}_{t\in\Delta\mathbb{Z}} \in \ell^1$. The sum on the right hand side is also continuous because

$$\sum_{k\in\mathbb{Z}} |\mathscr{G}(f+k/\Delta)| < \infty$$

because $\mathscr{G}$ is continuous and in L^1. Therefore, the result holds for all $f \in [-1/2\Delta, 1/2\Delta]$.