Time Series Exercise Sheet 5

Sofia Olhede

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Exercise 5.1

Suppose that $\{X_t\}$ is an ARIMA(p, d, q) process satisfying the equations

$$\phi(B)(\mathrm{I} - \mathrm{B})^d X_t = \theta(B) Z_t$$

where Z_t is white noise. Show that the same equations are also satisfied by $W_t = X_t + A_0 + A_1 t + \cdots + A_{d-1} t^{d-1}$, where A_j are independent random variables.

Solution 5.1

Firstly, if we apply the differencing operator to the zero sequence (i.e. $0 = \{0\}_{t\in\mathbb{Z}}$) then we have

$$(\mathrm{I} - \mathrm{B})0 = 0.$$

Furthermore, since the differencing operator is linear, we can consider the effect on each part of term in W_t, in other words

$$(\mathrm{I} - \mathrm{B})^d W_t = (\mathrm{I} - \mathrm{B})^d X_t + A_0(\mathrm{I} - \mathrm{B})^d 1 + A_1(\mathrm{I} - \mathrm{B})^d t + \cdots + A_{d-1}(\mathrm{I} - \mathrm{B})^d t^{d-1}.$$

Combining these two observations, we see that we need only show that

$$(\mathrm{I} - \mathrm{B})^k t^{k-1} = 0$$

for all $k \in \mathbb{N}$.

The proof of this can be achieved by induction. If $k = 1$ then we have

$$(\mathrm{I} - \mathrm{B})t^0 = (\mathrm{I} - \mathrm{B})1 = 0.$$

(Note here 0 and 1 refer to the zero and one sequences!)

Now assume that the statement holds for all $r \le k$ for some k. Then we have

$$(\mathrm{I} - \mathrm{B})^k t^{k-1} = 0.$$

Now we have

$$\begin{aligned}(\mathrm{I} - \mathrm{B})^{k+1} t^k &= (\mathrm{I} - \mathrm{B})^k (\mathrm{I} - \mathrm{B}) t^k \\ &= (\mathrm{I} - \mathrm{B})^k (t^k - (t-1)^k) \\ &= -(\mathrm{I} - \mathrm{B})^k \sum_{r=0}^{k-1} \binom{k}{r} t^r (-1)^{r-k}\end{aligned}$$

because the term involving t^k cancels out. Now by linearity, the induction hypothesis and that $(\mathrm{I} - \mathrm{B})0 = 0$, the result holds for $k + 1$. Therefore by induction, the result holds for all $k \in \mathbb{N}$.

Exercise 5.2

You observe U_t that follows a SARMA $(1,2)\times(0,1)_4$ model. Write down the equations that specify U_t in terms of the past. Determine the weights of the process in its infinite MA representation

$$U_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}.$$

You may find it useful to define ψ_j in terms of ψ_{j-1} as this will simplify algebra.

Solution 5.2

From the lecture we rewrite the given SARMA model as:

$$\begin{aligned}(1-\phi B)X_t &= (1-\theta_1 B - \theta_2 B^2)(1-\Theta B^4)\epsilon_t \Leftrightarrow \\ X_t &= \phi X_{t-1} + (1-\theta_1 B - \theta_2 B^2)(1-\Theta B^4)\epsilon_t.\end{aligned}$$

Recursively replacing X_{t-1} we may rewrite the above via an infinite series representation:

$$\begin{aligned}X_t &= \sum_{l=0}^{\infty} (\phi B)^l (1-\theta_1 B - \theta_2 B^2)(1-\Theta B^4)\epsilon_t \\ &= \sum_{l=0}^{\infty} (\phi^l B^l - \phi^l \theta_1 B^{l+1} - \phi^l \theta_2 B^{l+2} - \Theta\{\phi^l B^{l+4} - \phi^l \theta_1 B^{l+5} - \phi^l \theta_2 B^{l+6}\})\epsilon_t \\ &= \sum_{l=0}^{\infty} \psi_l \epsilon_{t-l}. \qquad (1)\end{aligned}$$

We start now finding the coefficients ψ_l corresponding to the first few terms of ϵ_{t-l}.

Clearly $\psi_0 = 1$. Similarly we find the following:

$$\psi_1 = -\theta_1 + \phi; \;\; \psi_2 = -\theta_2 - \theta_1\phi + \phi^2; \;\; \psi_3 = -\theta_2\phi - \theta_1\phi^2 + \phi^3 = \phi\psi_2.$$

Some care must be taken starting from ϵ_{t-4} as it is the first seasonal lag. Namely, proceeding similar to before, we compute:

$$\begin{aligned}\psi_4 &= -\Theta - \theta_2\phi^2 - \theta_1\phi^3 + \phi^4 = -\Theta + \phi\psi_3, \\ \psi_5 &= -\Theta(\phi - \theta_1) - \theta_2\phi^3 - \theta_1\phi^4 + \phi^5 \\ \psi_6 &= -\Theta(\phi^2 - \theta_1\phi - \theta_2) - \theta_2\phi^4 - \theta_1\phi^5 + \phi^6 = -\Theta(\phi^2 - \theta_1\phi - \theta_2) + (\phi^6 - \theta_1\phi^5 - \theta_2\phi^4) \\ \psi_7 &= -\Theta(\phi^3 - \theta_1\phi^2 - \theta_2\phi) + (\phi^7 - \theta_1\phi^6 - \theta_2\phi^5) = \phi\psi_6.\end{aligned}$$

Overall, we see a pattern arising from the recursion, which tells us that, for $l \geq 6$, ψ_l takes the form:

$$\psi_l = (\phi^l - \theta_1\phi^{l-1} - \theta_2\phi^{l-2}) - \Theta(\phi^{l-4} - \theta_1\phi^{l-5} - \theta_2\phi^{l-6}).$$

Exercise 5.3

Let us study the seasonal process $\left(1-0.7B^2\right)X_t = \left(1-0.3B^2\right)\epsilon_t$ for ϵ_t white noise with unit variance.

1. Find the coefficients $\{a_j\}$ in the representation $X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j}$
2. Find the coefficients $\{b_j\}$ in the representation $\epsilon_t = \sum_{j=0}^{\infty} b_j X_{t-j}$
3. Find and visualise the auto-correlation of X_t.

Solution 5.3

1. We are given $\left(1-\phi B^2\right) X_t = \left(1-\theta B^2\right)\epsilon_t$, where $\phi = 0.7$ and $\theta = 0.3$. As the polynomial $\Phi(z) = \left(1-\phi z^2\right)$ has no roots on the unit circle, X_t is stationary, and we may proceed as follows:

$$X_t = \frac{\left(1-\theta B^2\right)}{\left(1-\phi B^2\right)}\epsilon_t = \sum_{j=0}^{\infty}\left(\phi B^2\right)^j\left(1-\theta B^2\right)\epsilon_t \tag{2}$$

Similar to the previous exercise , we compute the coefficients a_j corresponding to ϵ_{t-j} :

$$a_0 = 1; a_1 = 0; a_2 = -\theta + \phi; a_3 = 0; \quad a_4 = -\phi\theta + \phi^2$$

Clearly, we see from eq. (2) that $a_j = 0$ for all odd j -s. Regarding j even we see that for $j \geq 2$ and even we have:

$$a_j = \phi^{j/2} - \phi^{\frac{j-2}{2}}\theta.$$

2. Similar to point a) we derive also in this case the representation:

$$\epsilon_t = \frac{\left(1-\phi B^2\right)}{\left(1-\theta B^2\right)}X_t = \sum_{j=0}^{\infty}\left(\theta B^2\right)^j\left(1-\phi B^2\right)X_t,$$

from where it follows that $b_0 = 1$ and $b_j = 0$ for j odd. Then for $j \geq 2$ and even we compute once again:

$$b_j = \theta^{j/2} - \theta^{\frac{j-2}{2}}\phi\ .$$

3. We want to compute ρ_τ. We proceed as follows:

$$\begin{aligned}
\operatorname{Cov}\left(X_t, X_{t+\tau}\right) &= \operatorname{Cov}\left(\sum_{j=0}^{\infty} a_j\epsilon_{t-j}, \sum_{k=0}^{\infty} a_k\epsilon_{t+\tau-k}\right)\\
&\overset{\text{Bounded Convergence}}{=} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_j a_k \operatorname{Cov}\left(\epsilon_{t-j}, \epsilon_{t+\tau-k}\right)\\
&\overset{\epsilon\text{ is white noise}}{=} \sigma_\epsilon^2\sum_{j=0}^{\infty} a_j a_{j+\tau}
\end{aligned}$$

and $\rho_\tau = \frac{\sum_{j=0}^{\infty} a_j a_{j+\tau}}{\sum_{j=0}^{\infty} a_j^2}$. We also compute $\operatorname{Var}\left(X_t\right) = \sigma_\epsilon^2\left\{1 + \left(1 - 2\phi^{-1}\theta + \phi^{-2}\theta^2\right)\frac{\phi^2}{1-\phi^2}\right\}$.

Clearly, for τ being odd, we see that $\rho_\tau = 0, \rho_0 = 1$, while for $\tau \geq 2$ and even we compute:

$$\begin{aligned}
\rho_\tau &= \frac{\sigma_\epsilon^2\left\{\phi^{\tau/2} - \phi^{\frac{\tau-2}{2}}\theta + \sum_{k=1}^{\infty}\left(\phi^{\frac{4k+\tau}{2}} - 2\phi^{\frac{4k+\tau-2}{2}}\theta + \phi^{\frac{4k+\tau-4}{2}}\theta^2\right)\right\}}{\operatorname{Var}\left(X_t\right)}\\
&= \frac{\sigma_\epsilon^2\left\{\phi^{\tau/2} - \phi^{\frac{\tau-2}{2}}\theta + \left(\phi^{\tau/2} - 2\phi^{\frac{\tau-2}{2}}\theta + \phi^{\frac{\tau-4}{2}}\theta^2\right)\sum_{k=1}^{\infty}\phi^{2k}\right\}}{\operatorname{Var}\left(X_t\right)}\\
&= \frac{\phi^{\tau/2} - \phi^{\frac{\tau-2}{2}}\theta + \frac{\left(\phi^{\tau/2} - 2\phi^{\frac{\tau-2}{2}}\theta + \phi^{\frac{\tau-4}{2}}\theta^2\right)\phi^2}{1-\phi^2}}{\left\{1 + \left(1 - 2\phi^{-1}\theta + \phi^{-2}\theta^2\right)\frac{\phi^2}{1-\phi^2}\right\}}\\
&= \phi^{\tau/2}\frac{1 - \phi^{-1}\theta + \frac{\left(1-2\phi^{-1}\theta+\phi^{-2}\theta^2\right)\phi^2}{1-\phi^2}}{\left\{1 + \left(1 - 2\phi^{-1}\theta + \phi^{-2}\theta^2\right)\frac{\phi^2}{1-\phi^2}\right\}}
\end{aligned}$$

The result is shown in Figure 1.

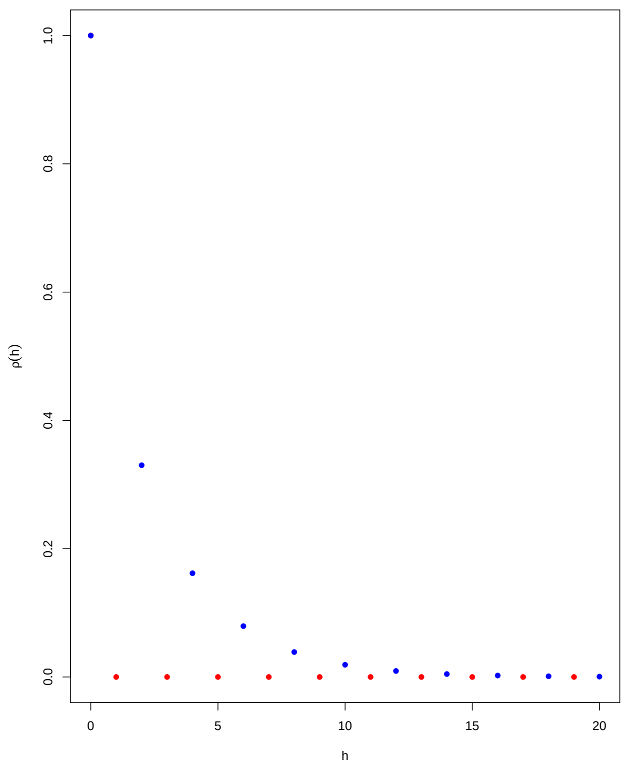

Figure 1: ρ_τ as a function of τ.