Time Series Exercise Sheet 3

Sofia Olhede

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Exercise 3.1

Find the coefficients ψ_j $j = 0, 1, 2, 3...$ in the representation of

$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j},$$

of the ARMA process

$$(1 - 0.5\mathrm{B} + 0.04\mathrm{B}^2)X_t = (1 + 0.25\mathrm{B})\varepsilon_t$$

when ε_t is white noise.

Solution 3.1

We have

$$\begin{aligned}
X_t &= \frac{1 + 0.25\mathrm{B}}{1 - 0.5\mathrm{B} + 0.04\mathrm{B}^2}\varepsilon_t \\
&= \frac{1 + 0.25\mathrm{B}}{0.04(\mathrm{B} - 2.5)(\mathrm{B} - 10)}\varepsilon_t \\
&= \frac{1 + 0.25\mathrm{B}}{0.04 \times 7.5}\left(\frac{-1}{\mathrm{B} - 2.5} + \frac{1}{\mathrm{B} - 10}\right)\varepsilon_t \\
&= \frac{1 + 0.25\mathrm{B}}{0.3}\left(\frac{1}{2.5}\sum_{j=0}^{\infty} 0.4^j \mathrm{B}^j - \frac{1}{10}\sum_{j=0}^{\infty} 0.1^j \mathrm{B}^j\right)\varepsilon_t \\
&= \sum_{j=0}^{\infty}\left\{\left(\frac{0.4^j}{0.75} - \frac{0.1^j}{3}\right)\mathrm{B}^j + \left(\frac{0.4^j}{3} - \frac{0.1^j}{12}\right)\mathrm{B}^{j+1}\right\}\varepsilon_j \\
&= \sum_{j=0}^{\infty}\left(\frac{0.4^j}{0.75} - \frac{0.1^j}{3}\right)\varepsilon_{t-j} + \sum_{l=1}^{\infty}\left(\frac{0.4^l}{1.2} - \frac{0.1^l}{1.2}\right)\varepsilon_{t-l} \\
&= \varepsilon_t + \sum_{j=1}^{\infty}\left(\frac{0.4^j}{0.75} - \frac{0.1^j}{3} + \frac{0.4^j}{1.2} - \frac{0.1^j}{1.2}\right)\varepsilon_{t-j} \\
&= \sum_{j=0}^{\infty}\psi_j \varepsilon_{t-j},
\end{aligned}$$

where

$$\psi_j = \begin{cases} 1 & \text{if } j = 0 \\ \frac{13}{6}0.4^j - \frac{7}{6}0.1^j & \text{if } j = 1, 2, \ldots \end{cases}$$

Exercise 3.2

Assume that Y_t is a causal and invertible ARMA process $\phi(\mathrm{B})Y_t = \theta(\mathrm{B})\epsilon_t$ define $\tilde{\phi}(\mathrm{B}) = \phi(\mathrm{B})\theta^{-1}(\mathrm{B})$ and take $a(\mathrm{B}) = (\tilde{\phi}(\mathrm{B}))^{-1} = \sum_{j=0}^{\infty} a_j \mathrm{B}^j$. Determine the representation of Y_t in terms of ϵ_t, firstly when Y_t is a stationary AR(1) and secondly when Y_t is a stationary AR(2).

Solution 3.2

We start with the AR(1) model $(1-\phi\mathrm{B})Y_t = \epsilon_t$. By causality, we may express Y_t as

$$Y_t = \frac{1}{1-\phi\mathrm{B}}\epsilon_t = \sum_{k=0}^{\infty}\phi^k\epsilon_{t-k}.$$

The AR(2) is slightly more complicated. Again we write $(1-\phi_1\mathrm{B}-\phi_2\mathrm{B}^2)Y_t = \epsilon_t$, where by stationarity and causality the roots of the polynomial $\Phi(z) = 1-\phi_1 z-\phi_2 z^2$ lie outside of the unit circle, that is $|z|>1$. We denote the two roots by g_1^{-1} and g_2^{-1}. We may then express the polynomial as $\Phi(z) = (1-g_1 z)(1-g_2 z)$.

Then it follows that

$$Y_t = \frac{1}{(1-g_1\mathrm{B})(1-g_2\mathrm{B})}\epsilon_t.$$

Manipulating the double series we see

$$\begin{aligned}
Y_t &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} g_1^{k_1} g_2^{k_2}\mathrm{B}^{k_1+k_2}\epsilon_t \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} g_1^{k_1} g_2^{k_2}\epsilon_{t-(k_1+k_2)} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} g_1^{k} g_2^{n-k}\epsilon_{t-n} && (n = k_1+k_2) \\
&= \sum_{n=0}^{\infty}\epsilon_{t-n}\sum_{k=0}^{n} g_1^{k} g_2^{n-k}.
\end{aligned}$$

In the third line above we use the fact that both $|g_1|, |g_2| < 1$, and hence we can apply the Cauchy Product formula for Series.

As a sum of a geometric series see that for $g_1 \neq g_2$

$$\sum_{k=0}^{n} g_1^k g_2^{n-k} = g_1^n \frac{1-\frac{g_2^{n+1}}{g_1^{n+1}}}{1-\frac{g_2}{g_1}} = \frac{g_1^{n+1}-g_2^{n+1}}{g_1-g_2}.$$

And thus we can write

$$Y_t = \frac{g_1}{g_1-g_2}\sum_{n=0}^{\infty} g_1^n\epsilon_{t-n} - \frac{g_2}{g_1-g_2}\sum_{n=0}^{\infty} g_2^n\epsilon_{t-n}.$$

One can then substitute $g_1^{-1} = \frac{-\phi_1-\sqrt{\phi_1^2+4\phi_2}}{2\phi_2}$, $g_2^{-1} = \frac{-\phi_1+\sqrt{\phi_1^2+4\phi_2}}{2\phi_2}$.

The case $g_1 = g_2$ then clearly $\sum_{k=0}^{n} g_1^k g_2^{n-k} = (n+1)g_1^n$.

Exercise 3.3

Assume we study an AR(2) process given by (where ϵ_ι is a Gaussian process):

$$\frac{15}{16}Y_t = \frac{1}{4}Y_{t-1} - \frac{1}{16}Y_{t-2} + \epsilon_t, \quad t = 0,1,2,3\ldots$$

Write down the characteristic AR polynomial associated with this process. Is this a stationary process? Is it invertible?

Solution 3.3

Rewriting our AR(2) process in terms of the backshift operator $Y_t\Phi(\mathrm{B}) = \epsilon_t$ we get:

$$Y_t(\frac{15}{16} - \frac{1}{4}\mathrm{B} + \frac{1}{16}\mathrm{B}^2) = \epsilon_t. \tag{1}$$

We know that for the AR process to be stationary the roots of the characteristic polynomial $\Phi(z)$ must lie outside of the unit circle. We find out that the roots of the characteristic polynomial given in (1) are $2+3.3166247903554i$ and $2-3.3166247903554i$, respectively, which both lie outside the unit circle and hence our process is stationary. An AR process is always invertible.

Exercise 3.4

Assume we study the ARMA$(2,1)$ process generated according to

$$Y_t - \frac{1}{4}Y_{t-1} + \frac{1}{16}Y_{t-2} = \epsilon_t - 0.5\epsilon_{t-1}, \quad t = 0, 1, 2, 3 \dots$$

Is this a stationary process? Is it invertible?

Solution 3.4

The ARMA$(2,1)$ process is defined as $Y_t - \frac{1}{4}Y_{t-1} + \frac{1}{16}Y_{t-2} = \epsilon_t - 0.5\epsilon_{t-1}$ for $t \in \mathbb{N}$. This can be written in terms of the backshift operator as $Y_t\left(1 - \frac{1}{4}\mathrm{B} + \frac{1}{16}\mathrm{B}^2\right) = \epsilon_t\,(1 - 0.5\mathrm{B})$, giving characteristic polynomials $\Phi(z) = z^2 - 4z + 16$ and $\Omega(z) = 1 - \frac{1}{2}z$ for which we get respectively roots $z = 2 \pm 2\sqrt{3}i$ and $z = 2$ which are both outside the unit circle. Thus the process is stationary and invertible.

Exercise 3.5

Let $\{U_t\}$ be a stationary zero-mean time series. Define

$$X_t = (1 - 0.4\mathrm{B})U_t$$

and

$$W_t = (1 - 2.5\mathrm{B})U_t.$$

(a) Express the autocorrelation functions of $\{X_t\}$ and $\{W_t\}$ in terms of that for $\{U_t\}$.

(b) Show that $\{X_t\}$ and $\{W_t\}$ have the same autocorrelation functions.

(c) Show that the process $\varepsilon_t = -\sum_{j=1}^{\infty}(0.4)^j X_{t+j}$ satisfies the difference equations

$$\varepsilon_t - 2.5\varepsilon_{t-1} = X_t.$$

Solution 3.5

1. We have $X_t = U_t - 0.4U_{t-1}$. Then for all $t, \tau \in T$ the autocovariance function is

$$\gamma_X(\tau) = \mathrm{Cov}\left(U_t - 0.4U_{t-1}, U_{t+\tau} - 0.4U_{t+\tau-1}\right) = 1.16\gamma_U(\tau) - 0.4\gamma_U(\tau-1) - 0.4\gamma_U(\tau+1)$$

so the corresponding autocorrelation function is

$$\rho_X(\tau) = \frac{\gamma_X(\tau)}{\gamma_X(0)} = \frac{1}{\gamma_X(0)}\left\{1.16\gamma_U(\tau) - 0.4\gamma_U(\tau-1) - 0.4\gamma_U(\tau+1)\right\}$$

Remember that $\gamma_U(\tau) = \rho_U(\tau)\gamma_U(0)$, so

$$\rho_X(\tau) = \frac{\gamma_U(0)}{\gamma_X(0)}\left\{1.16\rho_U(\tau) - 0.4\rho_U(\tau-1) - 0.4\rho_U(\tau+1)\right\}$$

Using the fact that the autocovariance function is symmetric, we have $\gamma_U(-1) = \gamma_U(1)$, so

$$\gamma_X(0) = 1.16\gamma_U(0) - 0.8\gamma_U(1) = \gamma_U(0)\left\{1.16 - 0.8\frac{\gamma_U(1)}{\gamma_U(0)}\right\} = \gamma_U(0)\left\{1.16 - 0.8\rho_U(1)\right\}$$

thus

$$\frac{\gamma_U(0)}{\gamma_X(0)} = \frac{1}{1.16 - 0.8\rho_U(1)}$$

Finally

$$\rho_X(\tau) = \frac{1}{1.16 - 0.8\rho_U(1)} \left\{1.16\rho_U(\tau) - 0.4\rho_U(\tau - 1) - 0.4\rho_U(\tau + 1)\right\}.$$

Similarly, we obtain

$$\rho_W(\tau) = \frac{1}{7.25 - 5\rho_U(1)} \left\{7.25\rho_U(\tau) - 2.5\rho_U(\tau - 1) - 2.5\rho_U(\tau + 1)\right\}$$

2. We note that

$$\rho_W(\tau) = \frac{6.25}{6.25\left\{1.16 - 0.8\rho_U(1)\right\}} \left\{1.16\rho_U(\tau) - 0.4\rho_U(\tau - 1) - 0.4\rho_U(\tau + 1)\right\} = \rho_X(\tau)$$

3. Direct calculation shows that if $\varepsilon_t = -\sum_{j=1}^{\infty} 0.4^j X_{t+j}$ then

$$\begin{aligned}
\varepsilon_t - 2.5\varepsilon_{t-1} &= -\sum_{j=1}^{\infty} 0.4^j X_{t+j} + 2.5\sum_{j=1}^{\infty} 0.4^j X_{t+j-1} \\
&= -\sum_{j=1}^{\infty} 0.4^j X_{t+j} + 2.5\sum_{l=0}^{\infty} 0.4^{l+1} X_{t+l} \\
&= -\sum_{j=1}^{\infty} 0.4^j X_{t+j} + X_t + \sum_{l=1}^{\infty} 0.4^l X_{t+l} \\
&= X_t
\end{aligned}$$