Time Series Exercise Sheet 1

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A note on notation: sometimes it is useful to discuss the autocovariance sequence of multiple different time series. Often, for a time series $\{X_t\}$, the autocovariance will be written as $\gamma_X(\tau)$, even if it is really a sequence. This is because the notation is usually cleaner than the alternatives such as $\gamma_{X,\tau}$. Whether the autocovariance is a sequence or a function will be clear from context.

Exercise 1.1

Let X_t be distributed as a student t distribution on one degree of freedom independently across t. Is $\{X_t\}$ weakly stationary?

Solution 1.1

Since X_t has 1 degree of freedom, its expectation and variance are undefined (X_t has a standard Cauchy distribution). Therefore, X_t is not weakly stationary.

Exercise 1.2

Let $\{X_t\}$ be second order stationary with autocovariance γ_τ and ε_t be iid with finite variance. Furthermore, assume $\{X_t\}$ and $\{\varepsilon_t\}$ are mutually independent. Determine the autocovariance of $Z_t = X_t + \varepsilon_t$.

Solution 1.2

Assuming that X_t and ε_t are independent, we have for all $t, \tau \in \mathbb{Z}$,

$$\mathrm{Cov}\left(X_t, \varepsilon_{t+\tau}\right) = \mathrm{Cov}\left(\varepsilon_t, X_{t+\tau}\right) = 0.$$

Moreover, since ε_t are iid we have $\mathrm{Cov}\left(\varepsilon_t, \varepsilon_{t+\tau}\right) = \mathrm{Var}\left(\varepsilon_t\right)$ if $\tau = 0$. If $\tau \neq 0$ then $\mathrm{Cov}\left(\varepsilon_t, \varepsilon_{t+\tau}\right)$ vanishes. Therefore,

$$\mathrm{Cov}\left(Z_t, Z_{t+\tau}\right) = \mathrm{Cov}\left(X_t, X_{t+\tau}\right) + \mathrm{Cov}\left(\varepsilon_t, \varepsilon_{t+\tau}\right) = \begin{cases} \mathrm{Var}\left(X_t\right) + \mathrm{Var}\left(\varepsilon_t\right) & \text{if } \tau = 0, \\ \gamma_\tau & \text{if } \tau \neq 0. \end{cases}$$

Exercise 1.3

Let Y_t be iid Gaussian random variables of mean 0 and variance σ_Y^2. Let a, b and c be constants. Which of the following processes are weakly stationary/strongly stationary, and if so, give their mean and ACVS.

1. $X_t = a + bY_t + cY_{t-1}$,
2. $X_t = a + bY_0$,
3. $X_t = Y_1 \cos(ct) + Y_2 \sin(ct)$
4. $X_t = Y_0 \cos(ct)$

Solution 1.3

1. We have $\mathrm{E}(X_t) = a + b\mathrm{E}(Y_t) + c\mathrm{E}(Y_{t-1}) = a$. Using the fact that Y_t are iid, we have

$$\begin{aligned}\operatorname{Cov}(X_t, X_{t+\tau}) &= b^2 \operatorname{Cov}(Y_t, Y_{t+\tau}) + bc \operatorname{Cov}(Y_t, Y_{t+\tau-1}) + bc \operatorname{Cov}(Y_{t-1}, Y_{t+\tau}) + c^2 \operatorname{Cov}(Y_{t-1}, Y_{t+\tau-1}) \\ &= \begin{cases} \left(b^2 + c^2\right)\sigma_Y^2 & \text{if } \tau = 0 \\ bc\sigma_Y^2 & \text{if } \tau = \pm 1 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

   Since $\operatorname{Cov}(X_t, X_{t+\tau})$ depends upon the lag τ only and not t, and the variance is finite, X_t is weakly stationary. Moreover, X_t is Gaussian, as a linear combination of Gaussian variables, so X_t is also strongly stationary.

2. Similarly, we have $\mathbb{E}[X_t] = a$, and $\operatorname{Cov}(X_t, X_{t+\tau}) = \operatorname{Cov}(bY_0, bY_0) = b^2\sigma_Y^2$. We conclude that X_t is weakly and strongly stationary.

3. We have $\mathbb{E}[X_t] = 0$, and

$$\begin{aligned}\operatorname{Cov}(X_t, X_{t+\tau}) &= \operatorname{Cov}(Y_1\cos(ct) + Y_2\sin(ct), Y_1\cos\{c(t+\tau)\} + Y_2\sin\{c(t+\tau)\}) \\ &= [\cos(ct)\cos\{c(t+\tau)\} + \sin(ct)\sin\{c(t+\tau)\}]\sigma_Y^2 \\ &= \cos\{ct - c(t+\tau)\}\sigma_Y^2 \\ &= \cos(c\tau)\sigma_Y^2\end{aligned}$$

   Thus, X_t is weakly and strongly stationary.

4. We have $\mathrm{E}(X_t) = 0$, and $\operatorname{Cov}(X_t, X_{t+\tau}) = \cos(ct)\cos\{c(t+\tau)\}\sigma_Y^2$. Therefore, X_t is not weakly stationary, so not strongly stationary.

Exercise 1.4

Determine the autocovariance of ϵ_t which is uncorrelated across t, but with fixed variance σ^2 and mean zero.

Solution 1.4

Clearly we have

$$\gamma_\varepsilon(\tau) = \begin{cases} 0 & \text{if } \tau \neq 0, \\ \sigma^2 & \text{if } \tau = 0. \end{cases}$$

Exercise 1.5

Determine the autocovariance of $X_t = \theta_1\epsilon_t + \theta_2\epsilon_{t-1}$ where ϵ_t which is mean zero, uncorrelated across t but with fixed variance σ^2.

Solution 1.5

We have

$$\begin{aligned}\gamma_X(\tau) &= \theta_1^2 \operatorname{Cov}(\varepsilon_t, \varepsilon_{t+\tau}) + \theta_1\theta_2 \operatorname{Cov}(\varepsilon_t, \varepsilon_{t+\tau-1}) + \theta_1\theta_2 \operatorname{Cov}(\varepsilon_{t-1}, \varepsilon_{t+\tau}) + \theta_2^2 \operatorname{Cov}(\varepsilon_{t-1}, \varepsilon_{t+\tau-1}) \\ &= \begin{cases} \left(\theta_1^2 + \theta_2^2\right)\sigma^2 & \text{if } \tau = 0 \\ \theta_1\theta_2\sigma^2 & \text{if } \tau = \pm 1 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Exercise 1.6

Suppose that $\{X_t\}$ and $\{Y_t\}$ are uncorrelated stationary time series, i.e. X_t and Y_s are uncorrelated for every choice of t and s. Show that the sequence $\{X_t + Y_t\}$ is stationary with an autocovariance that is the sum of the autocovariance sequences of $\{X_t\}$ and $\{Y_t\}$.

Solution 1.6

We have three conditions to check:

1. Since $\mathbb{E}[X_t]$ and $\mathbb{E}[Y_t]$ are constant (by the stationarity of X_t and Y_t), we have that $\mathbb{E}[X_t + Y_t] = \mathbb{E}[X_t] + \mathbb{E}[Y_t]$ is also constant.
2. For any $t \in \mathbb{Z}$, we have

$$\begin{aligned}\operatorname{Var}(X_t + Y_t) &= \operatorname{Var}(X_t) + 2\operatorname{Cov}(X_t, Y_t) + \operatorname{Var}(Y_t)\\ &= \operatorname{Var}(X_t) + \operatorname{Var}(Y_t)\\ &< \infty.\end{aligned}$$

3. Using the bilinearity of the covariance, we have for all $t, \tau \in T$

$$\begin{aligned}\operatorname{Cov}(X_t + Y_t, X_{t+\tau} + Y_{t+\tau}) &= \gamma_X(\tau) + \operatorname{Cov}(X_t, Y_{t+\tau}) + \operatorname{Cov}(Y_t, X_{t+\tau}) + \gamma_Y(\tau)\\ &= \gamma_X(\tau) + \gamma_Y(\tau).\end{aligned}$$

Therefore, $X_t + Y_t$ is also weakly stationary.

Exercise 1.7

Suppose that $\{X_{t,1}\}, \{X_{t,2}\}, \ldots, \{X_{t,m}\}$ are stationary processes with zero means and

$$\gamma_j(\tau) = \operatorname{Cov}(X_{t,j}, X_{t+\tau,j}).$$

If $\mathbb{E}[X_{t,j} X_{t+\tau,k}] = 0$ for all t, τ and $j \neq k$ determine the autocovariance sequence of

$$X_t = \sum_{j=1}^{m} X_{t,j}.$$

Solution 1.7

Using that the processes are zero mean, we have that

$$\operatorname{Cov}(X_{t,j}, X_{t+\tau,k}) = \mathbb{E}[X_{t,j} X_{t+\tau,k}] = \begin{cases}\gamma_j(\tau) & \text{if } j = k,\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

Using the fact that different processes X_t are not correlated and that they have zero means, we have for all $t, \tau \in T$

$$\begin{aligned}\gamma_X(\tau) &= \operatorname{Cov}\left(\sum_{j=1}^{m} X_{t,j}, \sum_{k=1}^{m} X_{t+\tau,k}\right) && \text{(by definition)}\\ &= \sum_{j=1}^{m}\sum_{k=1}^{m} \operatorname{Cov}(X_{t,j}, X_{t+\tau,k}) && \text{(by bilinearity of covariance)}\\ &= \sum_{j=1}^{m} \gamma_j(\tau). && \text{(by (1))}\end{aligned}$$

The easiest way to see why the double sum reduces to the single sum is to see the former as the sum of all the elements of a matrix that contains zero everywhere except on its diagonal.