Time Series Solutions to Mock Exam

1. (a) The time series $\{X_t\}$ is said to be second-order/weak or covariance stationary if for all $n \geq 1$ for any $t_1, \dots, t_n \in \mathbb{Z}$ and for all τ such that $t_1 + \tau, \dots, t_n + \tau \in \mathbb{Z}$ all the joint moments of order 1 and 2 of $X_{t_1}, \dots, X_{t_n}$ exist, are all finite and equal to the corresponding joint moments of $X_{t_1+\tau}, \dots, X_{t_n+\tau}$.

   Equivalently, $\forall t, s \in \mathbb{Z}$, $\forall \tau$ such that $t+\tau, s+\tau \in \mathbb{Z}$

   1. $\mathbb{E}[X_t] = \mu$,
   2. $\text{Var}(X_t) = \sigma^2 < \infty$,
   3. $\mathbb{E}[X_t X_{t+\tau}] = \mathbb{E}[X_s X_{s+\tau}]$.

   The time series $\{X_t\}$ is said to be completely/strong or strictly stationary if for all $n \geq 1$ for any $t_1, \dots, t_n \in \mathbb{Z}$ and for all τ such that $t_1 + \tau, \dots, t_n + \tau \in \mathbb{Z}$ the joint distribution of $X_{t_1}, \dots, X_{t_n}$ is the same as $X_{t_1+\tau}, \dots, X_{t_n+\tau}$.

   (b) An ARMA(p,q) process is specified by

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots \phi_p X_{t-p} + \epsilon_t - \theta_1 \epsilon_{t-1} - \dots - \theta_q \epsilon_{t-q},$$

   where ϕ_i and ϕ_i are constants for $i = 1, \dots, p$, and for $i = 1, \dots, p$, and $\{\epsilon_t\}$ is a white noise process.

   (c)

$$\begin{aligned}
\text{cov}(X_t, X_{t+\tau}) =& \mathbb{E}[X_t X_{t+\tau}] - \mathbb{E}[X_t]\,\mathbb{E}[X_{t+\tau}] \\
=& \mathbb{E}[(\epsilon_t - 0.2\epsilon_{t-1} + 0.1\epsilon_{t-2})(\epsilon_{t+\tau} - 0.2\epsilon_{t+\tau-1} + 0.1\epsilon_{t+\tau-2})] \\
=& \mathbb{E}[\epsilon_t\epsilon_{t+\tau} - 0.2\epsilon_t\epsilon_{t+\tau-1} + 0.1\epsilon_t\epsilon_{t+\tau-2} \\
& - 0.2\epsilon_{t-1}\epsilon_{t+\tau} + 0.04\epsilon_{t-1}\epsilon_{t+\tau-1} - 0.02\epsilon_{t-1}\epsilon_{t+\tau-2} \\
& + 0.1\epsilon_{t-2}\epsilon_{t+\tau} - 0.02\epsilon_{t-2}\epsilon_{t+\tau-1} + 0.01\epsilon_{t-2}\epsilon_{t+\tau-2}] \\
=& \mathbb{E}[\epsilon_t\epsilon_{t+\tau}] - 0.2\mathbb{E}[\epsilon_t\epsilon_{t+\tau-1}] + 0.1\mathbb{E}[\epsilon_t\epsilon_{t+\tau-2}] \\
& - 0.2\mathbb{E}[\epsilon_{t-1}\epsilon_{t+\tau}] + 0.04\mathbb{E}[\epsilon_{t-1}\epsilon_{t+\tau-1}] - 0.02\mathbb{E}[\epsilon_{t-1}\epsilon_{t+\tau-2}] \\
& + 0.1\mathbb{E}[\epsilon_{t-2}\epsilon_{t+\tau}] - 0.02\mathbb{E}[\epsilon_{t-2}\epsilon_{t+\tau-1}] + 0.01\mathbb{E}[\epsilon_{t-2}\epsilon_{t+\tau-2}] \\
=& \begin{cases} 0.1\sigma^2 & \text{if } \tau = -2, \\ -0.22\sigma^2 & \text{if } \tau = -1, \\ 1.05\sigma^2 & \text{if } \tau = 0, \\ -0.22\sigma^2 & \text{if } \tau = 1, \\ 0.1\sigma^2 & \text{if } \tau = 2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

2. (a) First note that the roots of both $\Phi(z) = \left(1 + 0.04^2 - 0.5z\right)$, and $\Theta(z) = (1 + 0.25z)$ lie outside of the unit circle, hence the process is both causal and invertible. For the moment define the new process $Z_t := (1 + 0.25B)\epsilon_t$, so that

$$\left(1 + 0.04^2 - 0.5B\right) X_t = Z_t.$$

To transform $\Phi(z)$ into a monic polynomial, we divide by $\left(1 + 0.04^2\right)$.
We re-write X_t as

$$\left(1 - \frac{0.5}{1 + 0.04^2}B\right) X_t = \frac{Z_t}{1 + 0.04^2} = \frac{1 + 0.25B}{1 + 0.04^2}\epsilon_t.$$

In its infinite series representation we get

$$X_t = \frac{1}{1 + 0.04^2}\frac{Z_t}{1 - \frac{0.5}{1+0.04^2}B} = \sum_{k=0}^{\infty}\left(\frac{0.5}{1 + 0.04^2}\right)^k \frac{1}{1 + 0.04^2} Z_{t-k}$$

We are not done yet, as $Z_t = (1 + 0.25B)\epsilon_t$; hence using this we compute,

$$\begin{aligned}
X_t &= \sum_{k=0}^{\infty}\left(\frac{0.5}{1 + 0.04^2}\right)^k \frac{1 + 0.25B}{1 + 0.04^2}\epsilon_{t-k} \\
&= \sum_{k=0}^{\infty}\left(\frac{0.5}{1 + 0.04^2}\right)^k \frac{1}{1 + 0.04^2}\epsilon_{t-k} + \sum_{k=0}^{\infty}\left(\frac{0.5}{1 + 0.04^2}\right)^k \frac{0.25}{1 + 0.04^2}\epsilon_{t-(k+1)} \\
&= \frac{\epsilon_t}{1 + 0.04^2} + \sum_{k=1}^{\infty}\left(\frac{0.5}{1 + 0.04^2}\right)^k \frac{1}{1 + 0.04^2}\epsilon_{t-k} + \sum_{k=0}^{\infty}\left(\frac{0.5}{1 + 0.04^2}\right)^k \frac{0.25}{1 + 0.04^2}\epsilon_{t-(k+1)} \\
&= \frac{\epsilon_t}{1 + 0.04^2} + \sum_{k=0}^{\infty}\left(\frac{0.5}{1 + 0.04^2}\right)^{k+1} \frac{1}{1 + 0.04^2}\epsilon_{t-(k+1)} + \sum_{k=0}^{\infty}\left(\frac{0.5}{1 + 0.04^2}\right)^k \frac{0.25}{1 + 0.04^2}\epsilon_{t-(k+1)} \\
&= \frac{\epsilon_t}{1 + 0.04^2} + \sum_{k=0}^{\infty}\left\{\left(\frac{0.5}{1 + 0.04^2}\right)^{k+1} \frac{1}{1 + 0.04^2} + \left(\frac{0.5}{1 + 0.04^2}\right)^k \frac{0.25}{1 + 0.04^2}\right\}\epsilon_{t-(k+1)}
\end{aligned}$$

and so $\lambda_0 = \frac{1}{1+0.04^2}$, and $\lambda_{k+1} = \left\{\left(\frac{0.5}{1+0.04^2}\right)^{k+1} \frac{1}{1+0.04^2} + \left(\frac{0.5}{1+0.04^2}\right)^k \frac{0.25}{1+0.04^2}\right\}$ for $\epsilon_{t-(k+1)}, k \geq 0$.

(b) The autocovariance function of the MA(∞)

$$
\begin{aligned}
\operatorname{cov}\left(X_t, X_{t+\tau}\right) &= \operatorname{cov}\left(\sum_{n_1=0}^{\infty} \lambda_{n_1} \epsilon_{t-n_1}, \sum_{n_2=0}^{\infty} \lambda_{n_2} \epsilon_{t+\tau-n_2}\right) \\
&= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \lambda_{n_1} \lambda_{n_2} \operatorname{cov}\left(\epsilon_{t-n_1}, \epsilon_{t+\tau-n_2}\right) \\
&= \sum_{n_1=0}^{\infty} \lambda_{n_1} \lambda_{n_1+\tau} \operatorname{var}\left(\epsilon_{t-n_1}\right) = \sum_{n_1=0}^{\infty} \lambda_{n_1} \lambda_{n_1+\tau} \sigma_\epsilon^2
\end{aligned}
$$

with the coefficients λ_{n_1} given in (a).
We have

$$
\lambda_{n_1} \lambda_{n_1+\tau} = \frac{1}{\left(1+0.04^2\right)^2}\left\{\left(\frac{0.5}{1+0.04^2}\right)^{2n_1+\tau+2} + 2\left(\frac{0.5}{1+0.04^2}\right)^{2n_1+\tau+1} 0.25 + \left(\frac{0.5}{1+0.04^2}\right)^{2n_1+2\tau} 0.25^2\right\}
$$

and so
$\sum_{n_1=0}^{\infty} \lambda_{n_1} \lambda_{n_1+\tau} \sigma_\epsilon^2 = \frac{\sigma_\epsilon^2}{(1+0.04^2)^2} \frac{1}{1-\frac{0.5^2}{\left(1+0.04^2\right)^2}} \left\{\left(\frac{0.5}{1+0.04^2}\right)^{\tau+2} + 0.5\left(\frac{0.5}{1+0.04^2}\right)^{\tau+1} + 0.25^2\left(\frac{0.5}{1+0.04^2}\right)^{2\tau}\right\}.$

(c) Method 1 Starting with $X_1, \ldots, X_n$, we may go backwards to estimate the error terms $\epsilon_1, \ldots, \epsilon_n$, as follows. For $t=0$ we let $X_0 = \epsilon_0 = 0$.
Then, $\epsilon_1 = \left(1+0.04^2\right) X_1 - 0.5 X_0 - 0.25\epsilon_0 = \left(1+0.04^2\right) X_1$. We then proceed recursively, to find $\epsilon_k = \left(1+0.04^2\right) X_k - 0.5 X_{k-1} - 0.25\epsilon_{k-1}$. Finally once we estimate $\epsilon_1, \ldots \epsilon_n$, we may predict X_{n+1} from the infinite series representation with coefficients in (a), and after truncating at $k=5$ we obtain

$$
\hat{X}_{n+1} = \mathbb{E}\left(\sum_{j=0}^{5} \lambda_k \epsilon_{n+1-k} \mid X_n, \ldots X_1\right) = \mathbb{E}\left(\sum_{j=1}^{5} \lambda_k \epsilon_{n+1-k} \mid X_n, \ldots X_1\right) = \sum_{j=1}^{5} \lambda_k \epsilon_{n+1-k}
$$

Method 2 For simplicity let $\phi_0 = \left(1+0.04^2\right), \phi_1 = 0.5, \theta = -0.25$. From here we find

$$
\begin{aligned}
&\epsilon_0 = 0 \\
&\epsilon_1 = \phi_0 X_1 \\
&\epsilon_2 = \phi_0 X_2 - \phi_1 X_1 + \theta_1 \epsilon_1 = \phi_0 X_2 - \left(\phi_1 - \theta_1 \phi_0\right) X_1 \\
&\epsilon_3 = \phi_0 X_3 - \phi_1 X_2 + \theta_1 \epsilon_2 = \phi_0 X_3 - \left(\phi_1 - \theta_1 \phi_0\right) X_2 - \theta_1\left(\phi_1 - \theta_1 \phi_0\right) X_1
\end{aligned}
$$

Now that we see a pattern we may proceed via induction and claim that

$$
\epsilon_k = \phi_0 X_k - \sum_{j=1}^{k-1} \theta_1^{j-1}\left(\phi_1 - \theta_1 \phi_0\right) X_{k-j}
$$

Clearly this is true for $k = 1, 2, 3$, so assume that it holds for some $k > 3$, and now focus on $k+1$.

We then get

$$\begin{aligned}
\epsilon_{k+1} &= \phi_0 X_{k+1} - \phi_1 X_k + \theta \epsilon_k \\
&= \phi_0 X_{k+1} - \phi_1 X_k + \theta \left(\phi_0 X_k - \sum_{j=1}^{k-1} \theta_1^{j-1} \left(\phi_1 - \theta_1 \phi_0\right) X_{k-j} \right) \\
&= \phi_0 X_{k+1} - \left(\phi_1 - \theta_1 \phi_0\right) X_k - \sum_{j=1}^{k-1} \theta_1^{j} \left(\phi_1 - \theta_1 \phi_0\right) X_{k-j} \\
&= \phi_0 X_{k+1} - \sum_{j=0}^{k-1} \theta_1^{j} \left(\phi_1 - \theta_1 \phi_0\right) X_{k-j} \\
&= \phi_0 X_{k+1} - \sum_{j=1}^{(k+1)-1} \theta_1^{j-1} \left(\phi_1 - \theta_1 \phi_0\right) X_{(k+1)-j}
\end{aligned}$$

Finally we we may predict X_{T+1} via,

$$\hat{X}_{T+1} = \mathbb{E}\left(X_{T+1} \mid X_T, \ldots X_1\right) = \frac{0.5 X_T + 0.25 \epsilon_T}{1 + 0.04^2}$$

where for ϵ_T we can use the formula above of ϵ_k.

3. (a) We first rewrite $\hat{S}_p(f)$ as

$$\hat{S}_p(f) = \frac{1}{N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} X_{t_1} X_{t_2} \exp\left\{-2i\pi f \left(t_1 - t_2\right)\right\}$$

Applying the expectation operator $\mathbb{E}$, and noting that $\mathrm{Cov}\left(X_{t_1}, X_{t_2}\right) = \mathbb{E}\left(X_{t_1} X_{t_2}\right)$ (mean is zero) we obtain

$$\mathbb{E}\left(\hat{S}_p(f)\right) = \frac{1}{N} \sum_{t=1}^{N} \mathbb{E}\left(X_t X_t\right) = \mathrm{Var}\left(X_t\right) = \sigma^2$$

(b) We rewrite $\hat{S}_p(f)$ again as

$$\hat{S}_p(f) = \frac{1}{N} \sum_{\tau=-N+1}^{N-1} \sum_{t=1}^{N-|\tau|} X_t X_{t+\tau} \exp\{-2i\pi f\tau\} = \sum_{\tau=-N+1}^{N-1} \hat{\gamma}_X^{(p)}(\tau) \exp\{-2i\pi f\tau\}$$

Integrating, and using linearity of the integral operator in the first passage, we compute

$$\begin{aligned}
\int_{-1/2}^{1/2} \hat{S}_p(f) df &= \sum_{\tau=-N+1}^{N-1} \int_{-1/2}^{1/2} \hat{\gamma}_X^{(p)}(\tau) \exp\{-2i\pi f\tau\} df \\
&= \sum_{\tau=-N+1}^{N-1} \hat{\gamma}_X^{(p)}(\tau) \int_{-1/2}^{1/2} \exp\{-2i\pi f\tau\} df \\
&= \sum_{\tau=-N+1}^{N-1} \hat{\gamma}_X^{(p)}(\tau) \int_{-1/2}^{1/2} \cos\{-2\pi f\tau\} df \\
&= \hat{\gamma}_X^{(p)}(0) + \sum_{\tau=-N+1\setminus\{0\}}^{N-1} \hat{\gamma}_X^{(p)}(\tau) \frac{\sin\{-\pi\tau\} - \sin\{\pi\tau\}}{-2\pi\tau} \\
&= \hat{\gamma}_X^{(p)}(0)
\end{aligned}$$

In the third passage we have used Euler's formula, and the fact that sine is an odd function.
Taking expectation we obtain,

$$\int_{-1/2}^{1/2} \hat{S}_p(f) df = \mathbb{E}\left(\hat{\gamma}_X^{(p)}(0)\right) = \mathbb{E}\left(\frac{1}{N} \sum_{t=1}^{N} X_t X_t\right) = \mathrm{Var}\left(X_t\right) = \sigma^2$$

(c) We have that $X_t = \epsilon_t + \theta\epsilon_{t-1}$ so therefore

$$\begin{aligned}
\mathrm{Cov}\left(X_{t+\tau}, X_t\right) &= \mathrm{Cov}\left(\epsilon_{t+\tau} + \theta\epsilon_{t+\tau-1}, \epsilon_t + \theta\epsilon_{t-1}\right) \\
&= \mathrm{Cov}\left(\epsilon_{t+\tau}, \epsilon_t\right) + \theta\,\mathrm{Cov}\left(\epsilon_{t+\tau}, \epsilon_{t-1}\right) + \theta\,\mathrm{Cov}\left(\epsilon_{t+\tau-1}, \epsilon_t\right) + \theta^2\,\mathrm{Cov}\left(\epsilon_{t+\tau-1}, \epsilon_{t-1}\right) \\
&= \sigma^2\delta_{\tau,0} + \theta\sigma^2\delta_{\tau+1,0} + \theta\sigma^2\delta_{\tau-1,0} + \theta^2\sigma^2\delta_{\tau,0} \\
&= \begin{cases} \sigma^2(1+\theta^2) & \text{if } \tau = 0, \\ \theta\sigma^2 & \text{if } |\tau| = 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This is invariant to t, and finite, and the mean is zero, hence the process is second-order stationary. Now the autocovariance of $\{X_t\}$ is therefore given by

$$\gamma_\tau = \begin{cases} \sigma^2(1+\theta^2) & \text{if } \tau = 0, \\ \theta\sigma^2 & \text{if } |\tau| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The spectral density function is then by definition given by

$$\begin{aligned}\mathscr{S}(f) &= \sum_{\tau=-\infty}^{\infty} \gamma_\tau e^{-2\pi i f \tau} \\ &= \theta\sigma^2 e^{2\pi i f} + (1+\theta^2)\sigma^2 + \theta\sigma^2 e^{-2\pi i f} \\ &= \sigma^2(1+\theta^2+2\theta\cos(2\pi f)).\end{aligned}$$

4. (a) Recall that $X_t^{(1)} = cX_{t-d}^{(2)} + \varepsilon_t$. We need to check three conditions for second-order stationarity.

Firstly, we need to check that the mean is constant over time. We have for all $t \in \mathbb{Z}$

$$\begin{aligned}\mathbb{E}\left[X_t^{(1)}\right] &= \mathbb{E}\left[cX_{t-d}^{(2)} + \varepsilon_t\right] \\ &= c\mathbb{E}\left[X_{t-d}^{(2)}\right] + \mathbb{E}\left[\varepsilon_t\right] \\ &= 0,\end{aligned}$$

by assumption.

Secondly, we need to check that the variance is finite and constant over time. We have for all $t \in \mathbb{Z}$

$$\begin{aligned}\operatorname{Var}\left(X_t^{(1)}\right) &= \operatorname{Var}\left(cX_{t-d}^{(2)} + \varepsilon_t\right) \\ &= c^2 \operatorname{Var}\left(X_{t-d}^{(2)}\right) + \operatorname{Var}\left(\varepsilon_t\right) \\ &< \infty,\end{aligned}$$

again because both $X_t^{(2)}$ and ε_t are assumed to be second-order stationary.

Finally, we need to check that the covariance is constant over time. We have for all $t, s, \tau \in \mathbb{Z}$

$$\begin{aligned}\operatorname{Cov}\left(X_t^{(1)}, X_{t+\tau}^{(1)}\right) &= \operatorname{Cov}\left(cX_{t-d}^{(2)} + \varepsilon_t, cX_{t+\tau-d}^{(2)} + \varepsilon_{t+\tau}\right) \\ &= c\operatorname{Cov}\left(X_{t-d}^{(2)}, X_{t+\tau-d}^{(2)}\right) + \operatorname{Cov}\left(\varepsilon_t, \varepsilon_{t+\tau}\right) \\ &= c\operatorname{Cov}\left(X_{s-d}^{(2)}, X_{s+\tau-d}^{(2)}\right) + \operatorname{Cov}\left(\varepsilon_s, \varepsilon_{s+\tau}\right) \\ &= \operatorname{Cov}\left(X_s^{(1)}, X_{s+\tau}^{(1)}\right),\end{aligned}$$

by stationarity of $X_t^{(2)}$ and ε_t.

(b) We already know that both $X_t^{(1)}$ and $X_t^{(2)}$ are second-order stationary, so we need only check the cross-covariance is invariant to shifts in time. We have for all $t, s, \tau \in \mathbb{Z}$

$$\begin{aligned}
\operatorname{Cov}\left(X_{t+\tau}^{(1)}, X_t^{(2)}\right) &= \operatorname{Cov}\left(cX_{t-d+\tau}^{(2)} + \varepsilon_t, X_t^{(2)}\right) \\
&= c\operatorname{Cov}\left(X_{t-d+\tau}^{(2)}, X_t^{(2)}\right) + \operatorname{Cov}\left(\varepsilon_t, X_t^{(2)}\right) \\
&= c\operatorname{Cov}\left(X_{s-d+\tau}^{(2)}, X_s^{(2)}\right) + 0 \\
&= \operatorname{Cov}\left(X_{s+\tau}^{(1)}, X_s^{(2)}\right),
\end{aligned}$$

where we used the fact that ε_t is uncorrelated with $X_t^{(2)}$, and the stationarity of $X_t^{(2)}$.

(c) Note that from the previous working (setting $s = 0$) we have

$$\begin{aligned}
\gamma_\tau^{(1,2)} &= c\gamma_{\tau-d}^{(2,2)} + 0 = c\gamma_{\tau-d}^{(2,2)} \\
\gamma_\tau^{(2,1)} &= \operatorname{Cov}\left(X_{t-\tau}^{(1)}, X_t^{(2)}\right) = c\gamma_{\tau-d}^{(2,2)} \\
\gamma_\tau^{(1,1)} &= c^2\gamma_\tau^{(2,2)} + \sigma_\epsilon^2\delta_{\tau,0}.
\end{aligned}$$

Write $S_{2,2}(f)$ for the spectral density of $X_t^{(2)}$ at $f \in \mathbb{R}$.

$$S_{2,2}(f) = \sum_{\tau=-\infty}^{\infty} \gamma_\tau^{(2,2)} e^{-2\pi i f\tau}$$

Then we have

$$\begin{aligned}
S_{1,2}(f) &= \sum_{\tau=-\infty}^{\infty} \gamma_\tau^{(1,2)} e^{-2\pi i f\tau} \\
&= \sum_{\tau=-\infty}^{\infty} c\gamma_{\tau-d}^{(2,2)} e^{-2\pi i f\tau} \\
&= c\sum_{\tau'=-\infty}^{\infty} \gamma_{\tau'}^{(2,2)} e^{-2\pi i f(\tau'+d)} && \text{(setting } \tau' = \tau - d\text{)} \\
&= ce^{-2\pi i f d} \sum_{\tau'=-\infty}^{\infty} \gamma_{\tau'}^{(2,2)} e^{-2\pi i f\tau'} \\
&= ce^{-2\pi i f d} S_{2,2}(f).
\end{aligned}$$

By symmetry we also have

$$\begin{aligned}
S_{2,1}\left(f\right) &= \sum_{\tau=-\infty}^{\infty} \gamma_{-\tau}^{(1,2)} e^{-2\pi i f \tau} \\
&= \sum_{\tau=-\infty}^{\infty} \gamma_{\tau'}^{(1,2)} e^{2\pi i f \tau'} && \text{(setting } \tau' = -\tau\text{)} \\
&= S_{2,1}\left(f\right)^{*}
\end{aligned}$$

$$ce^{2\pi i f d} S_{2,2}\left(f\right).$$

Finally, we have

$$\begin{aligned}
S_{1,1}\left(f\right) &= \sum_{\tau=-\infty}^{\infty} \gamma_{\tau}^{(1,1)} e^{-2\pi i f \tau} \\
&= \sum_{\tau=-\infty}^{\infty} \left(c^2 \gamma_{\tau}^{(2,2)} + \sigma_{\epsilon}^2 \delta_{\tau,0}\right) e^{-2\pi i f \tau} \\
&= c^2 \sum_{\tau=-\infty}^{\infty} \gamma_{\tau}^{(2,2)} e^{-2\pi i f \tau} + \sigma_{\epsilon}^2.
\end{aligned}$$