Ecole Polytechnique Federale de Lausanne

Month Year

MATH-342

Time Series

Date: xth of X, XXXX Time: Mock exam

Calculators may not be used. All questions should be answered.

1. (a) Please state (write down) the definitions of a stationary, as well as a strictly stationary process when time takes values in the index set $\mathbb{Z}$.

   (b) Define an Autoregressive Process with p terms and a Moving Average with q terms. Write down when such processes are stationary when $p = q = 1$.

   (c) Consider the zero-mean process

$$X_t = \epsilon_t - 0.2\epsilon_{t-1} + 0.1\epsilon_{t-2}, \qquad t \in \mathbb{Z}.$$

   Write down (calculate) the autocovariance sequence of X_t assuming $\mathrm{Var}(\epsilon_t) = \sigma^2 < \infty$ and $\mathrm{Cov}(\epsilon_{t+\tau}, \epsilon_t) = 0$ if $\tau \neq 0$.

2. (a) Find the coefficients λ_j for $j = 0, 1, 2, 3, \ldots$ if we calculate the representation of

$$X_t = \sum_{j=0}^{\infty} \lambda_j \epsilon_{t-j},$$

   for the ARMA process with backshift operator B if

$$(1 - 0.5B + 0.04^2)X_t = (1 + 0.25B)\epsilon_t.$$

   In this specification ϵ_t is white noise $\mathrm{Var}(\epsilon_t) = \sigma^2$.

   (b) Calculate the autocovariance sequence of X_t.

   (c) Assuming it is fine to truncate the infinite expansion at $j = 5$. Describe how to predict X_{n+1}, given you have observed $X_1, \ldots X_n$.

3. Define the periodogram estimator for a zero-mean stochastic process $\{X_t\}$ for a unit sampling rate to take the form

$$\widehat{S}^{(p)}(f) = \frac{1}{N}\left|\sum_{t=1}^{N} X_t e^{-2i\pi f t}\right|^2.$$

(a) Assume X_t is white noise, independent between observations and with variance $\mathrm{Var}(X_t) = \sigma^2$. Calculate $\mathbb{E}\{\widehat{S}^{(p)}(f)\}$.

(b) Calculate

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{S}^{(p)}(f)\, df, \tag{1}$$

in terms of the estimated autocovariance sequence, and take the expectation.

(c) Assume we observe $X_t = \epsilon_t + \theta\epsilon_{t-1}$ where ϵ_t is white noise, independent between observations and with variance $\mathrm{Var}(\epsilon_t) = \sigma^2$. Determine (and write down) the spectral density function of X_t.

4. Assume that we observe the delay process for $c \neq 0$ and $d \in \mathbb{Z}$:

$$X_t^{(1)} = cX_{t-d}^{(2)} + \varepsilon_t, \quad t \in \mathbb{Z}$$

where $X_t^{(2)}$ is second-order stationary and zero mean and ε_t is white noise, with variance $\mathrm{Var}\{\varepsilon_t\} = \sigma^2 < \infty$ and zero mean and $X_t^{(2)}$, ε_s are independent for all $t, s \in \mathbb{Z}$.

(a) Prove that $X_t^{(1)}$ is second-order stationary.

(b) Prove that the multivariate process $(X_t^{(1)}, X_t^{(2)})^T$ is jointly second-order stationary.

(c) Assuming that the autocovariance function of $X_t^{(2)}$ is given by $\gamma_\tau^{(2,2)}$ and

$$\sum_{\tau=-\infty}^{\infty} \left|\gamma_\tau^{(2,2)}\right| < \infty,$$

what are the spectral density functions and cross-spectral density functions of the process $(X_t^{(1)}, X_t^{(2)})^T$?