Time Series Exercise Sheet 10

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May 1, 2025

Exercise 10.1

Let ε_t be a white noise process. Show that the best one-step-ahead predictors for the causal $AR(2)$ process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

based on one observation X_1 and on two observations X_1, X_2 are

$$X_2^1 = \frac{\gamma_1}{\gamma_0} X_1 = \rho_1 X_1, \quad X_3^2 = \phi_1 X_2 + \phi_2 X_1.$$

Exercise 10.2

Theorem. *For a stationary process $\{X_t\}$, X_{n+h}^n is found by solving for $\beta_0, \ldots, \beta_n$ the prediction equations*

$$\mathbb{E}\left[\left(X_{n+h} - X_{n+h}^n\right) X_k\right] = 0, \quad k = 0, \ldots, n,$$

where $X_0 = 1$, and $X_{n+h}^n = \beta_0 + \sum_{j=1}^n \beta_j X_j$. Let $\mu = \mathbb{E}[X_t]$.

The β_j are then given by the solution of the system of equations

$$\boldsymbol{\Gamma}_n \boldsymbol{\beta} = \boldsymbol{\gamma}_{[h]}, \quad \beta_0 = \mu\left(1 - \boldsymbol{\beta}^{\mathrm{T}} \mathbf{1}_n\right)$$

with

$$\boldsymbol{\Gamma}_n = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n-1} & \gamma_{n-2} & \cdots & \gamma_0 \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}, \ \textit{and } \boldsymbol{\gamma}_{[h]} = \begin{pmatrix} \gamma_{n+h-1} \\ \gamma_{n+h-2} \\ \vdots \\ \gamma_h \end{pmatrix}. \tag{1}$$

1. Prove the above theorem
   (a) Express the mean prediction error of X_{n+h}^n in matrix form in terms of $\mathrm{Var}(\boldsymbol{X})$, $\boldsymbol{\beta}$ and β_0.
   (b) Find the expression of β_0 by using the properties of the best linear predictor.
   (c) Explain why there is no loss of generality in considering $\mu = 0$. What is then the value of β_0?
   (d) Show that if $\mu = 0$, then the best linear predictor satisfies
   $$\mathrm{Var}(\boldsymbol{X})\,\boldsymbol{\beta} = \mathrm{Cov}(X_{n+h}, \boldsymbol{X}). \tag{2}$$
   (e) Show that the prediction equations for $k = 1, \ldots, n$ are equivalent to (2). Explain how you found the prediction equation for $k = 0$ in the previous steps.
2. Assume $\boldsymbol{\Gamma}_n$ is invertible.
   (a) Express X_{n+h}^n in matrix form. How does it differ from the result seen in the lecture (e.g. $\mu = 0$)?
   (b) What is the prediction mean squared error? How does it differ from the result seen in the lecture?

Exercise 10.3

Prove the following theorem

Theorem. *The best linear predictor X_{n+h}^n for X_{n+h} in a causal ARMA process with general linear representation $\sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$ is*

$$X_{n+h}^n = \sum_{j=h}^{\infty} \psi_j \varepsilon_{n+h-j} = \psi_h \varepsilon_n + \psi_{h+1} \varepsilon_{n-1} + \cdots .$$

The corresponding prediction mean square error is $\sigma^2 \sum_{j=0}^{h-1} \psi_j^2$.