Time Series Exercise Sheet 9

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Exercise 9.1

For the process $\mathbf{Y}_t = \left(Y_t^{(1)},\, Y_t^{(2)}\right)^T$ generated by

$$Y_t^{(j)} = X_t^{(j)} + \varepsilon_t, \quad j = 1, 2$$

with $X^{(1)}$, $X^{(2)}$ and ε independent and where ε is white noise. Write

$$\sigma_j^2 = \mathrm{Var}\left(X_t^{(j)}\right)$$
$$\sigma_\epsilon^2 = \mathrm{Var}\left(\varepsilon_t\right)$$
$$\tilde{\gamma}_\tau^{(1,2)} = \mathrm{Cov}\left(X_{t+\tau}^{(1)}, X_t^{(2)}\right)$$

Use $\gamma_\tau^{(1,2)}$ to denote the cross-covariance sequence between $\left\{Y_t^{(1)}\right\}$ and $\left\{Y_t^{(2)}\right\}$. Calculate the correlation between $\left\{Y_t^{(1)}\right\}$ and $\left\{Y_t^{(2)}\right\}$ at all lags.

Exercise 9.2

For the delay process, with $\alpha > 0$ and $\nu \in \mathbb{Z}$

$$X_t^{(1)} = \alpha X_{t-\nu}^{(2)} + \varepsilon_t, \quad t \in \mathbb{Z}$$

where $\{\varepsilon_t\}$ is white noise and $\left\{X_t^{(2)}\right\}$ is stationary and uncorrelated with $\{\varepsilon_t\}$ at all lags

(a) Calculate the correlation between $\left\{X_t^{(1)}\right\}$ and $\left\{X_t^{(2)}\right\}$ at all lags.

(b) Calculate the cross spectrum, and determine the amplitude (absolute value) and phase (argument) of the cross-spectrum.

Exercise 9.3

For $\mathbf{Y}_t = \begin{pmatrix} Y_t^{(1)} \\ Y_t^{(2)} \end{pmatrix} \sim N(\mathbf{0}, \mathbf{\Sigma})$ independently in t with $\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$, determine the expectation and variance of $\widehat{\gamma}_\tau^{(1,2)}$ in

$$\widehat{\mathbf{\Gamma}}_\tau = \begin{bmatrix} \widehat{\gamma}_\tau^{(1,1)} & \widehat{\gamma}_\tau^{(1,2)} \\ \widehat{\gamma}_\tau^{(2,1)} & \widehat{\gamma}_\tau^{(2,2)} \end{bmatrix} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} \mathbf{Y}_{t+\tau} \mathbf{Y}_t^T$$