Time Series Exercise Sheet 8

Sofia Olhede

April 10, 2025

Exercise 8.1

Say that we observe X_t for $t \in T$ where $T = \{1, \ldots, n\}$. Recall that the periodogram is

$$\widehat{S}^{(p)}(f) = \sum_{\tau \in \mathbb{Z}} \hat{\gamma}_\tau e^{-2\pi i f \tau}. \tag{1}$$

Show that

$$\widehat{S}^{(p)}(f) = |J(f)|^2, \tag{2}$$

where

$$J(f) = \sqrt{\frac{1}{n}} \sum_{t \in T} \left(X_t - \bar{X}\right) e^{-2\pi i t f}. \tag{3}$$

Exercise 8.2

Recall that $u = g * h$ means

$$u_t = \sum_{s \in \mathbb{Z}} g_{t-s} h_s \tag{4}$$

for all $t \in \mathbb{Z}$ and $V = G * H$ means

$$V(f) = \int_{-1/2}^{1/2} G(f - f') H(f') \mathrm{d}f' \tag{5}$$

Assume that $g, h \in \ell^1$, prove the following:

(a) $h \cdot g \in \ell^1$ and $h * g \in \ell^1$,

(b) $H \in L^1$,

(c) If $v_t = h_t \cdot g_t$, then the Fourier transform of v is given by $V = H * G$,

(d) If $u = h * g$ then the Fourier transform of u is given by $U = H \cdot G$.

(Note, here L^1 refers to functions with domain $[-1/2, 1/2]$.)

Exercise 8.3

Replacing $\bar{X}$ with μ, show that

$$\mathbb{E}\left[\widehat{S}_h^{(p)}(f)\right] = \int_{-1/2}^{1/2} S(f') \left|H(f - f')\right|^2 \mathrm{d}f' \tag{6}$$

where H is the discrete Fourier transform of h.

Exercise 8.4

Consider the case that $\{X_t\}$ is a white noise process with variance σ^2. Prove that $\widehat{S}_h^{(p)}(f)$ is unbiased for all n if $\|h\|_2^2 = 1$.