Time Series Exercise Sheet 6

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Exercise 6.1

Let $g \in L^1$, have Fourier transform denoted by $\mathscr{G}$. Prove the following relations (where $\mathscr{H}$ is taken to be the Fourier transform of h).

1. If $h(t) = g(t)^*$ for all $t \in \mathbb{R}$ then
$$\mathscr{H}(f) = \mathscr{G}(-f)^*$$
for all $f \in \mathbb{R}$.
2. If $h(t) = g(\alpha t)$ for all $t \in \mathbb{R}$ then
$$\mathscr{H}(f) = \mathscr{G}(f/\alpha)/|\alpha|$$
for all $f \in \mathbb{R}$.
3. If $h(t) = g(t + \tau)$ for all $t \in \mathbb{R}$ then
$$\mathscr{H}(f) = \mathscr{G}(f)e^{2\pi i f \tau}$$
for all $f \in \mathbb{R}$.
4. Let $p \in L^1$ with Fourier transform $\mathscr{P}$. If $h(t) = \alpha g(t) + p(t)$ for all $t \in \mathbb{R}$ then
$$\mathscr{H}(f) = \alpha\mathscr{G}(f) + \mathscr{P}(f)$$
for all $f \in \mathbb{R}$.

Exercise 6.2

Prove the continuous time version of the convolution theorem, i.e. let $h, g \in L^1$, and let $u = g * h$, then
$$\mathscr{U} = \mathscr{G} \cdot \mathscr{H}.$$

Exercise 6.3

Prove the following theorem. Consider a continuous function $g \in L^1$. Then writing $\mathscr{G}$ for its Fourier transform and G for the Fourier transform of the sequence $\{g(t)\}_{t \in \Delta\mathbb{Z}}$ and assuming that $\mathscr{G} \in L^1$ we have
$$G(f) = \sum_{k \in \mathbb{Z}} \mathscr{G}(f + k/\Delta). \tag{1}$$