Time Series Exercise Sheet 2

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Exercise 2.1

When we defined the MA(q) process we specified $\theta_0 = -1$. In terms of the covariance structure for moving average processes, why do we not gain more generality by letting θ_0 be an arbitrary number?

Exercise 2.2

Determine that the moving average process defined by

$$X_t = \varepsilon_t - \theta\varepsilon_{t-1}$$

can be written as

$$X_t = \varepsilon_t - \sum_{j=1}^{p} \theta^j X_{t-j} - \theta^{p+1}\varepsilon_{t-p-1}$$

for any positive integer value of p.

Exercise 2.3

Assume that $Y_t = \sum_{s=-p}^{p} g_{s-t}\epsilon_s$ and $Z_t = \sum_{s=-p}^{p} h_{s-t}\epsilon_s$ where ϵ_t is zero-mean white noise. We define X_t pointwise by $Y_t + Z_t$. Determine the first and second moments of X_t.

Exercise 2.4

In the lectures we defined two different estimators for the ACVS:

$$\tilde{\gamma}_\tau = \frac{1}{n-|\tau|} \sum_{t=1}^{n-|\tau|} \left\{X_t - \bar{X}\right\}\left\{X_{t+|\tau|} - \bar{X}\right\},$$

$$\hat{\gamma}_\tau = \frac{1}{n} \sum_{t=1}^{n-|\tau|} \left\{X_t - \bar{X}\right\}\left\{X_{t+|\tau|} - \bar{X}\right\}.$$

If X_t has independent Gaussian realizations at each time point please calculate the mean, variance and mean-square error of these estimators. Repeat the calculation for an MA(1) process. As in class to do the analysis replace $\bar{X}$ by $\mu = \mathrm{E}(X_t)$, arguing that for large samples this will be appropriate.

(Hint): There is a known result (Isserlis' theorem) that if $\mathbf{X} = (X_1, X_2, X_3, X_4)$ is a multivariate normal random vector, then:

$$\mathbb{E}[X_1X_2X_3X_4] = \mathbb{E}[X_1X_2]\,\mathbb{E}[X_3X_4] + \mathbb{E}[X_1X_3]\,\mathbb{E}[X_2X_4] + \mathbb{E}[X_1X_4]\,\mathbb{E}[X_2X_3].$$

Exercise 2.5

Assume we know the mean of X_t and it is μ. Form the estimator

$$\bar{\gamma}_\tau^{(\alpha)} = \alpha\tilde{\gamma}_\tau + (1-\alpha)\hat{\gamma}_\tau$$

For X_t Gaussian white noise determine the MSE of $\bar{\gamma}_\tau^{(\alpha)}$. For $\sigma_X^2 = 1$, $\tau = 1$ and $n = 10$ plot it as a function of α. What value of α is appropriate?

Exercise 2.6

Show that for any series $\{x_1, \ldots, x_n\}$ the sample autocovariance satisfies $\sum_{|\tau|<n} \hat{\gamma}_\tau = 0$.