Recall that we proved last year that if $Y_1, ..., Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then:

- $\bar{Y} \sim N(\mu, \sigma^2)$
- $\bar{Y}$ is independent of S^2
- $\frac{n-1}{\sigma^2} S^2 \sim \chi^2_{n-1}$

I claimed during the lecture that this follows as a corollary of the Theorem in slide 76. To see this, let $Y_1, ..., Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$ and define

$$\varepsilon_i = Y_i - \mu$$

so that

$$Y_i = \mu + \varepsilon_i, \qquad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2).$$

This is starting to look like a linear model, albeit a very simple one with no explanatory variable and only an intercept parameter. Let's make this clear:

- we have the response vector $Y_{n\times 1} = (Y_1, ..., Y_n)^\top$
- we have the design matrix $X_{n\times 1} = (1, ..., 1)^\top$ (with only one column, i.e. $p = 1$).
- we have a 1×1 parameter vector $\beta = \mu$ (since $p = 1$ the parameter vector has only one entry, i.e. is scalar)
- and we have an error vector $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)^\top \sim N(0, \sigma^2 I_{n\times n})$

and so with this notation, it is clear that we have a linear model:

$$Y_i = \mu + \varepsilon_i, \qquad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2) \iff \underbrace{\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}}_{Y_{n\times 1}} = \underbrace{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}_{X_{n\times 1}} \underbrace{\mu}_{\beta_{1\times 1}} + \underbrace{\begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}}_{\varepsilon_{n\times 1}}, \quad \underbrace{\begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}}_{\varepsilon_{n\times 1}} \sim N(0, \sigma^2 I_{n\times n}).$$

It remains to notice that

$$\hat{\beta} = (X^\top X)^{-1} X^\top Y = \left(\sum_{i=1}^n 1 \cdot 1 \right)^{-1} \sum_{i=1}^n 1 \cdot Y_i = \bar{Y}$$

and that (since $p = 1$)

$$S^2 = \frac{1}{n-1} \|Y - X\hat{\beta}\|^2 = \frac{1}{n-1} \left\| \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} - \begin{pmatrix} \bar{Y} \\ \vdots \\ \bar{Y} \end{pmatrix} \right\|^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2$$

and to then apply the theorem in slide 76.