Definition. A group Γ (without topology) is called **residually finite** if for every non-trivial $\gamma \in \Gamma$ there is a homomorphism $f\colon \Gamma \to F$ to a finite group F with $f(\gamma) \neq e$.

Exercise 1. Let Γ be a residually finite group.

(i) Prove that for every finite set $S \subseteq \Gamma$ there is a homomorphism $f\colon \Gamma \to F$ to a finite group F which is injective on S.

(ii) Prove that Γ can be embedded as a subgroup into a compact topological group.

(iii) If you contemplate your proof of (ii), can you describe *intrinsically* the induced topology obtained on Γ? That is, try to give a fundamental system (=basis) of neighbourhoods of e in Γ without referring to the compact group that you constructed for (ii).

Exercise 2. Let $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Compute $\exp A$, $\exp B$, $\exp A \exp B$ and $\exp(A+B)$.

Hint: the first computations should really be super-easy if you start by looking at X^n for $X = A, B$ in the definition of $\exp X$. For $X = A+B$, you need to be a little bit more organized and use the definition of (the real number) $e = \exp 0$.

Exercise 3. Let $d \in \mathbf{N}$ and let A, B be $d \times d$-matrices over $\mathbf{R}$. Give a precise proof of the following statements:

(i) The series $\exp A$ converges, i.e. the sequence of partial sums converges in the space $\mathbf{M}_d(\mathbf{R})$ of $(d \times d)$-matrices.

(ii) If $AB = BA$, then $\exp(A+B) = \exp A \exp B$. In particular, $\exp A$ is invertible for all A.

(iii) Compute the derivative at $X = 0$ of the map exp. Make sure first that you understand the domain and range of that map to see what format the derivative has.