Exercise 1. Fix an integer $n \geq 2$. Since $G = \mathrm{GL}_n(\mathbf{R})$ is an open set in $\mathbf{R}^{n^2}$, we can view $C_c(G)$ in $C_c(\mathbf{R}^{n^2})$ (check this). In particular, for any $f \in C_c(G)$ we can consider the Lebesgue measure:

$$\lambda(f) = \int_{\mathbf{R}^{n^2}} f(g)\, dg_{1,1} dg_{1,2} \cdots dg_{1,n} dg_{2,1} \cdots dg_{n,n}.$$

where $g_{i,j}$ are the coefficients of the matrix g.

(i) For $n = 2$, write this out nicely, without any "$\cdots$".
(ii) Prove that this measure λ on G is not left invariant and not right invariant.
(iii) Define a left invariant (non-zero) measure on G.

Hints. You can start by doing everything for $n = 2$ and without indices, writing $g = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$ etc. For (ii), it is possible to do this without computing the measure of any function, just by examining what matrix multiplication does, especially scalar matrices. For (iii), it helps to remember the change of variables formula from Analysis II.

Notation and definitions. Let $(V, \langle \cdot, \cdot \rangle)$ be a Hilbert space and write $\mathscr{L}(V)$ for the space of all continuous linear maps $\alpha \colon V \to V$. Seeing α in V^V, we can endow $\mathscr{L}(V)$ with the product of the norm topology on each factor V; this is called the **strong operator topology** (SOT). For instance, $\alpha_n \to \alpha$ means simply $\|\alpha(v) - \alpha(v)\| \to 0$ for all $v \in V$.
Likewise, the **weak operator topology** (WOT) is the product of the weak topology on each factor V.

Exercise 2. Prove that SOT and WOT agree on the orthogonal group $\mathrm{O}(V) \subseteq \mathscr{L}(V)$. First, give an example to show that SOT$\neq$WOT in general.

Hint: use Ex. 2 from last week.

Exercise 3. Prove that $\mathrm{O}(V)$ is a topological group for SOT and WOT.

Hint: use Ex. 2 from this week.