Exercise 1. Let X be a compact (Hausdorff) space and recall that **probability** measures are the normalised, positive measures:

$$\mathrm{Prob}(X) = \{\mu \in C(X)' : \mu(1_X) = 1, \mu \geq 0\}.$$

(i) Give all the details of the proof that $\mathrm{Prob}(X)$ is weak-* compact.
(ii) Prove that the unit sphere in $C(X)'$ is, in general, not weak-* compact.

Hints. For (ii), you can choose e.g. $X = [0,1]$ *or* $X = \{0,1\}^{\mathbf{N}}$ *or whatever is easiest for you. But you cannot choose* $X = \{0,1\}$*; why?*

Exercise 2. Let V be a Hilbert space; you can take $V = \ell^2(\mathbf{N})$ if you like.
Prove that the weak and norm topology agree on the unit sphere $S \subseteq V$. First, give an example where this fails for the unit ball of V.

Hints. In $V = \ell^2(\mathbf{N})$*, let* $\delta_n \in V$ *be the sequence which is* 1 *at* n *and* 0 *otherwise. Consider the sequence (of sequences!)* $\{\delta_n\}_{n\in\mathbf{N}}$*. Meditate why the first and second part of the exercise do not contradict each other.*

Exercise 3. Consider the following topological group G. As a space, $G = \mathbf{R} \times \mathbf{R}_{>0}$ with the usual topology. The group operation is

$$(a,b)(a',b') = (a + ba', bb').$$

(This is an example of a **semidirect product** and is usually denoted by $\mathbf{R} \rtimes \mathbf{R}_{>0}$.)

(i) Check that there are embeddings of $\mathbf{R}$ (additive) and $\mathbf{R}_{>0}$ (multiplicative) into G. There is also a quotient homomorphism $G \to \mathbf{R}_{>0}$. All this becomes even easier if you realize that G is isomorphic to a certain subgroup of $\mathrm{GL}_2(\mathbf{R})$.

(ii) Give an explicit formula for a (non-zero) invariant measure on the multiplicative group $\mathbf{R}_{>0}$.

(iii) Find a (non-zero) left invariant measure on G. Is your measure right invariant?

Hints. For (iii), we really want an explicit formula that associates a number to any given $f \in C_c(G)$ *given in the coordinates* (a,b)*; use part (ii). Then, just compute your formula for any* $f \in C_c(G)$ *and then for a right translate of* f*. You can take* f *as super-simple as you want (just not* $f = 0$*).*