For the first three exercises, let $V \neq 0$ be a vector space and let $\|\cdot\|$ be a norm on V.

Exercise 1. The norm induces a topology on V defined by taking open balls as a sub-base.

(i) Check that V is a topological group for the addition. *Remark: you might also check that the scalar multiplication $\mathbf{R} \times V \to V$ is continuous; thus V is a* ***topological vector space****.*

(ii) Prove that a linear map $\lambda\colon V \to \mathbf{R}$ is continuous if and only if it is bounded on the closed unit ball of V.

(iii) Let V' be the collection of all continuous linear maps $\lambda\colon V \to \mathbf{R}$ (the **topological dual** of V). Check that V' is a vector space and prove that

$$\|\lambda\| = \sup_{v \in V \smallsetminus \{0\}} \frac{|\lambda(v)|}{\|v\|}$$

defines a norm on V'.

(iv) Given $v \in V$, define $\hat{v} \in V''$ by $\hat{v}(\lambda) = \lambda(v)$. Check that $v \mapsto \hat{v}$ is a well-defined continuous linear map. If you know the Hahn–Banach theorem, then you can check that this map preserves the norm and in particular it is injective.

Exercise 2. (i) The **weak topology** on V is defined by taking all sets $\lambda^{-1}(U)$ as a sub-base, where $\lambda \in V'$ and $U \subseteq \mathbf{R}$ is open. Prove that V is a topological group (or topological vector space) for this topology.

(ii) The **weak-* topology** on V' is defined by taking all sets $\hat{v}^{-1}(U)$ as a sub-base, where $v \in V$ and $U \subseteq \mathbf{R}$ is open. Prove that V' is a topological group (or topological vector space) for this topology.

Exercise 3. Prove that the closed unit ball[1] of V' is compact for the weak-* topology.

Hint: You can embed V' into a huge product of copies of $\mathbf{R}$ in such a way that the unit ball lands in a product of compact intervals.

Side-question: so, do you think that V' is a locally compact group for the weak-* topology?

Exercise 4. Let $G = \prod_{i \in I} F_i$ be the product of a family of finite groups F_i (each F_i endowed with the discrete topology). Without using the theorem from the lecture, prove that G admits a Haar measure.

Hint: As a warm-up, make sure you understand the case where I is finite. For I infinite, explain first why it is enough to define the measure on a dense subspace of $C(G)$. Then, choose such a subspace by thinking about the case of finite products.

[1] "closed" might sound ambiguous; this always means $\{\lambda : \|\lambda\| \leq 1\}$. Is it weak-* closed?