Exercise 0. Show that a projective limit of connected topological spaces can be disconnected.
Hint: you can work with easy subsets of $\mathbf{R}^2$*...*

Exercise 1. Consider the compact circle group $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ and the group homomorphism f that I call "squaring the circle"(!), namely: $f(z) = z^2$. Let S be the projective limit of the resulting projective system

$$S^1 \xleftarrow{f} S^1 \xleftarrow{f} S^1 \xleftarrow{f} S^1 \xleftarrow{f} \cdots$$

Prove that S is a connected group. Can you see whether S is locally connected?

Exercise 2. The goal of this exercise is to prove that *every projective limit of connected compact groups is connected (and compact).* So, Ex. 0 was a warning and Ex. 1 was a warming!

(i) Recall why compactness of the limit holds.

(ii) Let G be a Hausdorff topological group and suppose that $\mathscr{H}$ is a projective system of compact connected subgroups $H < G$ indexed by inclusion. This means:

$$\forall H_1, H_2 \in \mathscr{H} \; \exists H \in \mathscr{H} : H_1, H_2 \supseteq H.$$

Explain why this is indeed a special case of a projective system: what is the index set, what are the maps, what is the limit. Prove that the limit is connected.
Hint: wlog, G *is compact and hence* T_4.

(iii) Show that every projective limit can be obtained by the situation descibed in (ii); deduce the general statement of the exercise.

Exercise 3. Let p be a prime and consider the profinite group G defined by the inverse system

$$\mathbf{Z}/p\mathbf{Z} \longleftarrow \mathbf{Z}/p^2\mathbf{Z} \longleftarrow \cdots \longleftarrow \mathbf{Z}/p^n\mathbf{Z} \longleftarrow \mathbf{Z}/p^{n+1}\mathbf{Z} \longleftarrow \cdots$$

where every arrow is the reduction modulo the corresponding power of p. Prove that G is isomorphic to $\mathbf{Z}_p$ as a topological group.

Hints. Use the universal property to obtain a nice homomorphism $\mathbf{Z} \to G$*. Then compare the topology that this induces on* $\mathbf{Z}$ *with the* p*-adic topology studied in Problem Sets 3 and 4.*