Exercise 0. Check that you understood all the basic properties of the "bracket" $(f : F)$.

Exercise 1. Let G be a locally compact group and let μ be a left Haar measure on G. Given a function $f \in C_c(G)$, define $\check{f}$ by $\check{f}(x) = f(x^{-1})$. Define further $\check{\mu}$ by $\check{\mu}(f) = \mu(\check{f})$.

(i) Check that $\check{\mu}$ is a measure (according to our definition as continuous linear functional).
(ii) Prove something usefuly about the invariance of $\check{\mu}$...

Exercise 2. Let G be a locally compact group, μ a left Haar measure on G and $f \in C_c(G)$. Prove that the function

$$G \longrightarrow \mathbf{R}, \quad g \longmapsto \mu(R_g f)$$

is continuous.

Hints. Using left uniform continuity (Problem Set 10), observe that $g \mapsto R_g f$ is continuous for the sup-norm. This almost finishes the exercise if you note that we can assume $\mu \geq 0$ and $f \geq 0$.

Bonus question: try to give a precise meaning (and then a proof) to the statement that the norm-continuity of $g \mapsto R_g f$ is actually *equivalent* to left uniform continuity.

Exercise 3. In Ex. 3 of Problem Set 8, you found some (non-zero) left invariant measure on $G = \mathbf{R} \rtimes \mathbf{R}_{>0}$. Let's denote your measure by μ.

(i) For $g \in G$ and $f \in C_c(G)$, give a formula for $\mu(R_g f)$, where we recall that $(R_g f)(x) = f(xg)$.
(ii) Verify that your formula gives in particular a continuous homomorphism $\Delta \colon G \to \mathbf{R}_{>0}$.
(iii) Find a (non-zero) right invariant measure ν on G.
(iv) Give a formula for $\nu(L_g f)$, recalling that $(L_g f)(x) = f(g^{-1}x)$.