EXERCISES FOR RANDOMIZATION AND CAUSATION (MATH-336)

Exercise Sheet 2

Exercise 1 (Conditional independence. Inspired by Jamie Robins' lectures)**.** Prove the following identities for independence, assuming that X, Y, Z, W are discrete:

(a) Symmetry: $X \perp\!\!\!\perp Y \mid Z \Longleftrightarrow Y \perp\!\!\!\perp X \mid Z$.
(b) Decomposition: $X \perp\!\!\!\perp Y, W \mid Z \Longrightarrow X \perp\!\!\!\perp Y \mid Z$.
(c) Weak Union: $X \perp\!\!\!\perp Y, W \mid Z \Longrightarrow X \perp\!\!\!\perp Y \mid Z, W$.
(d) Contraction: $(X \perp\!\!\!\perp W \mid Y, Z)$ and $(X \perp\!\!\!\perp Y \mid Z) \Longrightarrow (X \perp\!\!\!\perp Y, W \mid Z)$.

Solution:

(a)

$$\begin{aligned} X \perp\!\!\!\perp Y \mid Z \Longleftrightarrow f(x \mid z) f(y \mid z) &= f(x, y \mid z) \\ &= f(y \mid z) f(x \mid z) \Longleftrightarrow Y \perp\!\!\!\perp X \mid Z \ . \end{aligned}$$

(b)

$$X \perp\!\!\!\perp Y, W \mid Z \Longleftrightarrow f(x, y, w \mid z) = f(x \mid z) f(y, w \mid z) \ ,$$

so summing over w gives

$$f(x, y \mid z) = f(x \mid z) f(y \mid z) \Longleftrightarrow X \perp\!\!\!\perp Y \mid Z \ .$$

(c) $X \perp\!\!\!\perp Y, W \mid Z$ implies that

$$f(x, y, w \mid z) = f(x \mid z) f(y, w \mid z) \ .$$

Multiplying both sides by $\frac{f(z)}{f(w,z)}$, we find that

$$f(x, y \mid w, z) = f(x \mid z) f(y \mid w, z) \ .$$

Next, decomposition of $X \perp\!\!\!\perp Y, W \mid Z$ implies $X \perp\!\!\!\perp W \mid Z$ and therefore

$$f(x, y \mid w, z) = f(x \mid w, z) f(y \mid w, z) \Longleftrightarrow X \perp\!\!\!\perp Y \mid W, Z \ .$$

(d) $X \perp\!\!\!\perp W \mid Y, Z$ implies that

$$\begin{aligned} f(x, w \mid y, z) &= f(x \mid y, z) f(w \mid y, z) \\ &\overset{X \perp\!\!\!\perp Y \mid Z}{=} f(x \mid z) f(w \mid y, z) \ . \end{aligned}$$

Next, expanding LHS as $f(x, w \mid y, z)$ as $f(x \mid w, y, z) f(w \mid y, z)$, we get that

$$f(x \mid w, y, z) = f(x \mid z) \Longleftrightarrow X \perp\!\!\!\perp Y \mid Z \ .$$

$\square$

Exercise 2 (Crossover experiments. Based on Fine Point 2.1 and Fine Point 3.2 [1]). In crossover experiments, individuals are observed during two or more periods. For simplicity, consider two periods, $t = 0$ and $t = 1$. An individual receives a different treatment A_t in each period t. Let $Y_1^{a_0,a_1}$ be the *deterministic* counterfactual outcome[1] at $t = 1$ if the individual is treated with $A_0 = a_0$ at $t = 0$ and $A_1 = a_1$ at $t = 1$. Let $Y_0^{a_0}$ be defined similarly for $t = 0$. The individual causal effect $Y_t^{a_t=1} - Y_t^{a_t=0}$ can be identified if the following three conditions hold:

i) no carryover effect of treatment: $Y_{t=1}^{a_0,a_1} = Y_{t=1}^{a_1}$ for all $a_0 \in \{0, 1\}$,
ii) the individual causal effect is constant in time: $Y_t^{a_t=1} - Y_t^{a_t=0} = \alpha$ for all $t \in \{0, 1\}$, and
iii) the counterfactual outcome under no treatment does not depend on time: $Y_t^{a_t=0} = \beta$ for all $t \in \{0, 1\}$.

Here, α and β are random variables that may differ between individuals.

Answer the following:

(a) Do any of the conditions i)-iii) hold by design in a randomized trial?
(b) For each of the conditions i)-iii), suggest a situation where the condition fails.
(c) Suppose individuals in the study are assigned to one of two crossover treatment regimes: $(A_0, A_1) = (0, 1)$ or $(1, 0)$. By assuming conditions i)-iii), show that the identification formula for the *individual* causal effect $Y_t^{a_t=1} - Y_t^{a_t=0}$ at all $t \in \{0, 1\}$ in terms of observed outcomes Y_t is

$$\alpha = (Y_1 - Y_0)A_1 + (Y_0 - Y_1)A_0 \ .$$

(d) Suppose that i)-ii) hold but iii) is violated and that $Y_1^{a_1=0} - Y_0^{a_0=0} = R$. Show that under randomization of treatments A_t, where individuals are randomized to either $(A_0, A_1) = (0, 1)$ with probability 1/2 or $(A_0, A_1) = (1, 0)$ with probability 1/2, the *average* causal effect $E[Y_t^{a_t=1} - Y_t^{a_t=0}]$ at all times $t \in \{0, 1\}$ is identified by

$$E[\alpha] = E[(Y_1 - Y_0)A_1 + (Y_0 - Y_1)A_0] \ .$$

(e) Suppose now that treatments are randomized as in (d), but that only condition i) holds. Let $\alpha_t = Y_t^{a_t=1} - Y_t^{a_t=0}$ for $t \in \{0, 1\}$. Show that the time-average of the average causal effect is identified by

$$\frac{1}{2}\left(E[\alpha_0] + E[\alpha_1]\right) = E\left[(Y_0 - Y_1)A_0 + (Y_1 - Y_0)A_1\right] \ .$$

Solution:

(a) No, conditions i)-iii) are properties of the treatment A and outcome Y that are not guaranteed to hold by randomly assigning A.
(b) Condition i) fails if the treatment has long-term effects on the outcome. For example, sustained smoking increases the risk of lung cancer even after the individual has quit smoking. ii) fails if Y is tumor growth, A is a chemotherapeutic drug and the tumor becomes resistant to the drug after some time. iii) fails if Y is an indicator function of survival at time t, and A is aspirin in patients with coronary artery disease. Because treatment with aspirin reduces the risk of heart attacks in these patients, treatment also increases the chances of survival.

[1] Some authors denote the counterfactuals by $Y_i^{a_0}$, that is, using subscripts i, when discussing individuals causal effects, to highlight that $Y_i^{a_0}$ may differ between individuals. To simplify the notation, we have omitted the subscript.

(c) An individual can either receive treatment $(A_0, A_1) = (0,1)$ or $(1,0)$, as the combinations $(1,1)$ and $(0,0)$ are, by design, not assigned. The identification formula for the individual causal effect will be different in each case:

$$\begin{aligned}
\alpha &\overset{\text{ii)}}{=} Y_t^{a_t=1} - Y_t^{a_t=0} \text{ for all } t \in \{0,1\} \\
&= Y_0^{a_0=1} - Y_0^{a_0=0} \\
&= Y_0^{a_0=1} - Y_1^{a_1=0} - \underbrace{\left(Y_0^{a_0=0} - Y_1^{a_1=0}\right)}_{\overset{\text{iii)}}{=}\beta-\beta=0} \\
&\overset{\text{i)}}{=} Y_0^{a_0=1} - Y_1^{a_0=1,a_1=0} \\
&\overset{\text{consistency}}{=} Y_0 - Y_1 \quad \text{if} \quad (A_0, A_1) = (1,0) \ .
\end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned}
Y_t^{a_t=1} - Y_t^{a_t=0} &\overset{\text{ii)}}{=} Y_1^{a_1=1} - Y_1^{a_1=0} \\
&= Y_1^{a_1=1} - Y_0^{a_0=0} - \underbrace{\left(Y_1^{a_1=0} - Y_0^{a_0=0}\right)}_{\overset{\text{iii)}}{=}\beta-\beta=0} \\
&\overset{\text{i)}}{=} Y_1^{a_0=0,a_1=1} - Y_0^{a_0=0} \\
&\overset{\text{consistency}}{=} Y_1 - Y_0 \quad \text{if} \quad (A_0, A_1) = (0,1) \ .
\end{aligned} \tag{2}$$

We get the identification formula by combining Eqs. 1 and 2, letting A_0 and A_1 play the role of indicator functions.

(d) Proceeding as in part (c),

$$\begin{aligned}
\alpha &\overset{\text{ii)}}{=} Y_0^{a_0=1} - Y_0^{a_0=0} \\
&= Y_0^{a_0=1} - Y_1^{a_1=0} - \underbrace{\left(Y_0^{a_0=0} - Y_1^{a_1=0}\right)}_{=-R} .
\end{aligned}$$

By consistency,

$$Y_0^{a_0=1} - Y_1^{a_1=0} = Y_0 - Y_1 \quad \text{if} \quad (A_0, A_1) = (1,0) \ ,$$

and thus have that

$$\alpha = Y_0 - Y_1 + R \quad \text{if} \quad (A_0, A_1) = (1,0) \ . \tag{3}$$

Likewise,

$$\begin{aligned}
\alpha &\overset{\text{ii)}}{=} Y_1^{a_1=1} - Y_1^{a_1=0} \\
&= Y_1^{a_1=1} - Y_0^{a_1=0} - \underbrace{\left(Y_1^{a_1=0} - Y_0^{a_0=0}\right)}_{=R}
\end{aligned}$$

and we therefore have

$$\alpha = Y_1 - Y_0 - R \quad \text{if} \quad (A_0, A_1) = (0,1) \ . \tag{4}$$

Then,

$$E[(Y_0 - Y_1)A_0 + (Y_1 - Y_0)A_1] = E[(Y_0 - Y_1)A_0] + E[(Y_1 - Y_0)A_1] \ .$$

We evaluate each of the terms on the RHS separately,

$$\begin{aligned} E[(Y_0 - Y_1)A_0] &= E\{E[(Y_0 - Y_1)A_0 \mid A_0)]\} \\ &= E\{A_0 \cdot E[(Y_0 - Y_1) \mid A_0)]\} \\ &= \sum_{a_0} p(a_0)a_0 \sum_{y_0}\sum_{y_1} p(y_0, y_1 \mid a_0)(y_0 - y_1) \\ &= P(A_0 = 1)E[Y_0 - Y_1 | A_0 = 1] \\ &\overset{\text{Eq. 3}}{=} P(A_0 = 1)(E[\alpha] - E[R]) \ . \end{aligned}$$

Using analogous arguments, we can show that

$$E[(Y_1 - Y_0)A_1] \overset{\text{Eq. 4}}{=} P(A_1 = 1)(E[\alpha] + E[R]).$$

Because $P(A_0 = 1) = P(A_1 = 1) = \frac{1}{2}$, we obtain the desired identification formula.

(e) Arguing as in (d), we find that

$$\begin{aligned} \alpha_0 &= Y_0 - Y_1 + R \quad \text{if} \quad (A_0, A_1) = (1, 0) \ , \\ \alpha_1 &= Y_1 - Y_0 - R \quad \text{if} \quad (A_0, A_1) = (0, 1) \ . \end{aligned}$$

The final result follows by an argument similar to the one in part (d).

Exercise 3 (Collapsibility and odds ratios. Based on Fine Point 4.3 [1, 2, 3])**.** Consider a randomized $A \in \{0, 1\}$, assigned by flipping an unbiased coin, and outcome $Y \in \{0, 1\}$. Suppose there exist subgroups (for example women and men) defined by the covariate $V \in \{0, 1\}$ with positivity for A, i.e. satisfying

(5) $$P(A = a \mid V = v) > 0 \text{ for all } a \in \{0, 1\} \text{ whenever } P(V = v) > 0 \ .$$

(a) Does $Y^a \perp\!\!\!\perp A$ hold? Does $Y^a \perp\!\!\!\perp A \mid V = v$ hold for all $v = 0, 1$?

(b) Using the exchangeability condition $Y^a \perp\!\!\!\perp A$, prove that the following causal (counterfactual) estimand within subgroups, $P(Y^a = y \mid V = v)$, is identified by

$$P(Y^a = y \mid V = v) = P(Y = y \mid A = a, V = v) \ .$$

(c) By rewriting the marginal relative risk (RR) as a weighted average of the conditional relative risks (RR_v), prove that any probability law $P(A = a, V = v, Y = y)$ satisfying the positivity conditions

$$P(Y = 1 \mid A = 0) > 0$$

and Eq. 5 also satisfies

$$RR \in \left[\min_v(RR_v), \max_v(RR_v)\right] \ ,$$

where

$$RR = \frac{P(Y^{a=1} = 1)}{P(Y^{a=0} = 1)}$$

and

$$RR_v = \frac{P(Y^{a=1} = 1 \mid V = v)}{P(Y^{a=0} = 1 \mid V = v)} \ .$$

In other words, the marginal risk ratio lies in the range of the conditional relative risk ratios.

(d) Show also that the marginal risk difference $RD = P(Y^{a=1} = 1) - P(Y^{a=0} = 1)$ lies in the range of the conditional risk differences

$$RD_v = P(Y^{a=1} = 1 \mid V = v) - P(Y^{a=0} \mid V = v)$$

under the positivity condition in Eq. 5.

(e)* Find an example of a law $P(A = a, Y = y, V = v)$ such that

$$OR_{v=1} = OR_{v=0} > OR,$$

where

$$OR_v = \frac{P(Y^{a=1} = 1 \mid V = v)}{P(Y^{a=1} = 0 \mid V = v)} \Big/ \frac{P(Y^{a=0} = 1 \mid V = v)}{P(Y^{a=0} = 0 \mid V = v)}$$

and

$$OR = \frac{P(Y^{a=1} = 1)}{P(Y^{a=1} = 0)} \Big/ \frac{P(Y^{a=0} = 1)}{P(Y^{a=0} = 0)} \ .$$

Present your answer in the form of a table with entries $A \times Y \times V$. Deduce that in general we cannot write OR as a weighted sum of OR_v with non-negative weights.

This property is referred to as the non-collapsibility of the odds ratio, and can be seen as a consequence of Jensen's inequality (an average over a non-linear function does not equal the function evaluated on the average). Thus, reporting odds ratios as effect measures arguably has undesirable features.

Solution:

(a) Because A is randomly assigned (unconditionally), $Y^a \perp\!\!\!\perp A$ and $Y^a \perp\!\!\!\perp A \mid V$.[2]

(b) Using part (a),

$$\begin{aligned} P(Y^a = y \mid V = v) &\overset{Y^a \perp\!\!\!\perp A \mid V}{=} P(Y^a = y \mid A = a, V = v) \\ &= P(Y = y \mid A = a, V = v) \ , \end{aligned}$$

where we also used consistency and positivity.

(c) Using weights $w(v) = P(Y = 1 \mid A = 0, V = v)P(V = v)$, which are well-defined by positivity,

$$RR = \frac{\sum_v P(Y = 1 \mid A = 1, V = v)P(V = v)}{\sum_{v'} P(Y = 1 \mid A = 0, V = v')P(V = v')} = \sum_v RR_v \cdot \frac{w(v)}{\sum_{v'} w(v')} \ .$$

The marginal risk ratio lies in the range of the conditional risk ratios because any linear combination of RR_v with non-negative weights lies in $[\min_v(RR_v), \max_v(RR_v)]$. □

(d) By taking the sum $\sum_v P(V = v) \times (\cdot)$ over the identification formula in part (a), and using identical reasoning to part (b), we get the corresponding result for the risk difference. □

[2] As we will see later in the course, $A \perp\!\!\!\perp V \mid C$ for *any* pre-treatment variable C, because C occurs prior to A and therefore cannot be a collider on the path $A \to C \leftarrow Y^a$.

(e)* An example of the desired probability law is shown in Table 1.

TABLE 1. Probability law of A, Y, V demonstrating non-collapsibility of the odds ratio

	$V=1$		$V=0$		Marginal	
	$A=1$	$A=0$	$A=1$	$A=0$	$A=1$	$A=0$
$Y=1$	0.2	0.15	0.1	0.05	0.3	0.2
$Y=0$	0.05	0.1	0.15	0.2	0.2	0.3
Risks	0.8	0.6	0.4	0.2	0.6	0.4
Risk differences	0.2		0.2		0.2	
Risk ratios	1.33		2		1.5	
Odds ratios	2.67		2.67		2.25	

The fact that

$$OR \notin \left[\min_v(OR_v), \max_v(OR_v)\right]$$

implies that OR cannot be expressed as a weighted average of OR_v.

Exercise 4 (Positivity for standardization and IPW. Based on Technical Point 3.1 [1]). Consider a binary treatment A, a discrete baseline covariates L and an outcome Y. In the derivation of the weighted identification formula in the lectures, we showed that causal effect could expressed as the contrast

$$E[Y^{a=1} - Y^{a=0}] = E\left[\frac{I(A=1)}{\pi[A \mid L]}Y\right] - E\left[\frac{I(A=0)}{\pi[A \mid L]}Y\right] \tag{6}$$

under the assumption of conditional exchangeability, consistency. and positivity. The positivity condition is

$$P(A=a \mid L=l) > 0 \text{ for all } a \in \{0,1\} \text{ whenever } P(L=l) > 0\ .$$

Next, we will consider what happens when positivity is violated. Suppose that there exists some a^*, l such that $P(A=a^* \mid L=l) = 0$ and $P(L=l) > 0$. Next, define $Q(a) = \{l : P(A=a \mid L=l) > 0\}$ to be the levels of L with positivity for treatment level a.

(a) Show that

$$E\left[\frac{I(A=a)Y}{\pi[A \mid L]}\right] = P(L \in Q(a)) \sum_{l \in Q(a)} E[Y \mid A=a, L=l] P(L=l \mid L \in Q(a))\ .$$

(b) Explain why the naive contrast $E\left[\frac{I(A=1)}{\pi[A|L]}Y\right] - E\left[\frac{I(A=0)}{\pi[A|L]}Y\right]$ no longer has a causal interpretation under violation of positivity.

Solution:

(a) Firstly, notice that $\pi[A \mid L]$ cannot be zero,[3] even under violation of positivity when $\pi[a \mid L]$ *is* zero. To see this, recall that $\pi[A \mid L]$ is a random variable obtained by evaluating the conditional distribution function $\pi[a \mid l] := P(A = a \mid L = l)$ at the random arguments A and L, i.e. for all $\omega \in \Omega$, and thus $\pi[A, L](\omega) = \pi[A(\omega) \mid L(\omega)]$.

Therefore, events of the form $\{A = a^*, L = l\}$ do not contribute to the expectation in Eq. 6. This allows us to rewrite Eq. 6 as

$$E\left[\frac{I(A=a)Y}{\pi[A \mid L]}\right] = \sum_{\tilde{a}ly} P(Y = y, L = l, A = \tilde{a}) \frac{yI(\tilde{a} = a)}{\pi[\tilde{a} \mid l]}$$

$$= \sum_{\tilde{a}} \sum_{y} \sum_{l \in Q(a)} P(Y = y, L = l, A = \tilde{a}) \frac{yI(\tilde{a} = a)}{\pi[\tilde{a} \mid l]} \ .$$

Next, we can expand the joint density as

$$P(Y = y, L = l, A = \tilde{a}) = P(Y = y \mid A = \tilde{a}, L = l)P(A = \tilde{a} \mid L = l)P(L = l)$$

and thus obtain

$$\begin{aligned}
E\left[\frac{I(A=a)Y}{\pi[A \mid L]}\right] &= \sum_{\tilde{a}} \sum_{y} \sum_{l \in Q(a)} yI(a = \tilde{a}) \frac{1}{P(A = \tilde{a} \mid L = l)} \\
&\quad \times P(Y = y \mid A = \tilde{a}, L = l)P(A = \tilde{a} \mid L = l)P(L = l) \\
&= \sum_{y} \sum_{l \in Q(a)} yP(Y = y \mid A = a, L = l)P(L = l) \\
&= \sum_{l \in Q(a)} E[Y \mid A = a, L = l]P(L = l) \\
&= \sum_{l \in Q(a)} E[Y \mid A = a, L = l]P(L = l, L \in Q(a)) \\
&= P(L \in Q(a)) \sum_{l \in Q(a)} E[Y \mid A = a, L = l]P(L = l \mid L \in Q(a)) \ .
\end{aligned}$$

(b) When positivity fails, $Q(0) \neq Q(1)$, and therefore the contrast $E\left[\frac{I(A=1)}{\pi[A|L]}Y\right] - E\left[\frac{I(A=0)}{\pi[A|L]}Y\right]$ compares different groups of individuals. Conversely, $Q(0) = Q(1)$ when positivity holds, and therefore the contrast is the average causal effect if exchangeability holds.

References

[1] Miguel Hernan and James M. Robins. *Causal Inference*. Chapman & Hall, 2018.

[2] Sander Greenland and Judea Pearl. Adjustments and their Consequences-Collapsibility Analysis using Graphical Models: Adjustments and their Consequences. *International Statistical Review*, 79(3):401–426, December 2011.

[3] Strictly, $\pi[A \mid L]$ is almost surely non-negative. The events where positivity fails are measure zero events, $\mathbb{P}(\{\omega : A(\omega) = a^*, L(\omega) = l\}) = 0$, because

$$0 = \pi[a^* \mid l] := \mathbb{P}(A = a^* \mid L = l) := \frac{\mathbb{P}(\{\omega : A(\omega) = a^*, L(\omega) = l\})}{\mathbb{P}(\{\omega : L(\omega) = l\})}$$

and the denominator is clearly non-negative.

[3] Judea Pearl, James M. Robins, and Sander Greenland. Confounding and Collapsibility in Causal Inference. *Statistical Science*, 14(1):29–46, February 1999.