EXERCISES FOR RANDOMIZATION AND CAUSATION (MATH-336)

Exercise Sheet 2

Exercise 1 (Conditional independence. Inspired by Jamie Robins' lectures)**.** Prove the following identities for independence, assuming that X, Y, Z, W are discrete:

(a) Symmetry: $X \perp\!\!\!\perp Y \mid Z \Longleftrightarrow Y \perp\!\!\!\perp X \mid Z$.
(b) Decomposition: $X \perp\!\!\!\perp Y, W \mid Z \Longrightarrow X \perp\!\!\!\perp Y \mid Z$.
(c) Weak Union: $X \perp\!\!\!\perp Y, W \mid Z \Longrightarrow X \perp\!\!\!\perp Y \mid Z, W$.
(d) Contraction: $(X \perp\!\!\!\perp W \mid Y, Z)$ and $(X \perp\!\!\!\perp Y \mid Z) \Longrightarrow (X \perp\!\!\!\perp Y, W \mid Z)$.

Exercise 2 (Crossover experiments. Based on Fine Point 2.1 and Fine Point 3.2 [1])**.** In crossover experiments, individuals are observed during two or more periods. For simplicity, consider two periods, $t = 0$ and $t = 1$. An individual receives a different treatment A_t in each period t. Let $Y_1^{a_0,a_1}$ be the *deterministic* counterfactual outcome[1] at $t = 1$ if the individual is treated with $A_0 = a_0$ at $t = 0$ and $A_1 = a_1$ at $t = 1$. Let $Y_0^{a_0}$ be defined similarly for $t = 0$. The individual causal effect $Y_t^{a_t=1} - Y_t^{a_t=0}$ can be identified if the following three conditions hold:

i) no carryover effect of treatment: $Y_{t=1}^{a_0,a_1} = Y_{t=1}^{a_1}$ for all $a_0 \in \{0, 1\}$,
ii) the individual causal effect is constant in time: $Y_t^{a_t=1} - Y_t^{a_t=0} = \alpha$ for all $t \in \{0, 1\}$, and
iii) the counterfactual outcome under no treatment does not depend on time: $Y_t^{a_t=0} = \beta$ for all $t \in \{0, 1\}$.

Here, α and β are random variables that may differ between individuals.

Answer the following:

(a) Do any of the conditions i)-iii) hold by design in a randomized trial?
(b) For each of the conditions i)-iii), suggest a situation where the condition fails.
(c) Suppose individuals in the study are assigned to one of two crossover treatment regimes: $(A_0, A_1) = (0, 1)$ or $(1, 0)$. By assuming conditions i)-iii), show that the identification formula for the *individual* causal effect $Y_t^{a_t=1} - Y_t^{a_t=0}$ at all $t \in \{0, 1\}$ in terms of observed outcomes Y_t is

$$\alpha = (Y_1 - Y_0)A_1 + (Y_0 - Y_1)A_0 \ .$$

(d) Suppose that i)-ii) hold but iii) is violated and that $Y_1^{a_1=0} - Y_0^{a_0=0} = R$. Show that under randomization of treatments A_t, where individuals are randomized to either $(A_0, A_1) = (0, 1)$ with probability 1/2 or $(A_0, A_1) = (1, 0)$ with probability 1/2, the *average* causal effect $E[Y_t^{a_t=1} - Y_t^{a_t=0}]$ at all times $t \in \{0, 1\}$ is identified by

$$E[\alpha] = E[(Y_1 - Y_0)A_1 + (Y_0 - Y_1)A_0] \ .$$

[1]Some authors denote the counterfactuals by $Y_i^{a_0}$, that is, using subscripts i, when discussing individuals causal effects, to highlight that $Y_i^{a_0}$ may differ between individuals. To simplify the notation, we have omitted the subscript.

(e) Suppose now that treatments are randomized as in (d), but that only condition i) holds. Let $\alpha_t = Y_t^{a_t=1} - Y_t^{a_t=0}$ for $t \in \{0,1\}$. Show that the time-average of the average causal effect is identified by

$$\frac{1}{2}\left(E[\alpha_0] + E[\alpha_1]\right) = E\left[(Y_0 - Y_1)A_0 + (Y_1 - Y_0)A_1\right] \ .$$

Exercise 3 (Collapsibility and odds ratios. Based on Fine Point 4.3 [1, 2, 3])**.** Consider a randomized $A \in \{0,1\}$, assigned by flipping an unbiased coin, and outcome $Y \in \{0,1\}$. Suppose there exist subgroups (for example women and men) defined by the covariate $V \in \{0,1\}$ with positivity for A, i.e. satisfying

$$P(A = a \mid V = v) > 0 \text{ for all } a \in \{0,1\} \text{ whenever } P(V = v) > 0 \ . \tag{1}$$

(a) Does $Y^a \perp\!\!\!\perp A$ hold? Does $Y^a \perp\!\!\!\perp A \mid V = v$ hold for all $v = 0, 1$?

(b) Using the exchangeability condition $Y^a \perp\!\!\!\perp A$, prove that the following causal (counterfactual) estimand within subgroups, $P(Y^a = y \mid V = v)$, is identified by

$$P(Y^a = y \mid V = v) = P(Y = y \mid A = a, V = v) \ .$$

(c) By rewriting the marginal relative risk (RR) as a weighted average of the conditional relative risks (RR_v), prove that any probability law $P(A = a, V = v, Y = y)$ satisfying the positivity conditions

$$P(Y = 1 \mid A = 0) > 0$$

and Eq. 1 also satisfies

$$RR \in \left[\min_v(RR_v), \max_v(RR_v)\right] \ ,$$

where

$$RR = \frac{P(Y^{a=1} = 1)}{P(Y^{a=0} = 1)}$$

and

$$RR_v = \frac{P(Y^{a=1} = 1 \mid V = v)}{P(Y^{a=0} = 1 \mid V = v)} \ .$$

In other words, the marginal risk ratio lies in the range of the conditional relative risk ratios.

(d) Show also that the marginal risk difference $RD = P(Y^{a=1} = 1) - P(Y^{a=0} = 1)$ lies in the range of the conditional risk differences

$$RD_v = P(Y^{a=1} = 1 \mid V = v) - P(Y^{a=0} \mid V = v)$$

under the positivity condition in Eq. 1.

(e)* Find an example of a law $P(A = a, Y = y, V = v)$ such that

$$OR_{v=1} = OR_{v=0} > OR,$$

where

$$OR_v = \frac{P(Y^{a=1} = 1 \mid V = v)}{P(Y^{a=1} = 0 \mid V = v)} \Big/ \frac{P(Y^{a=0} = 1 \mid V = v)}{P(Y^{a=0} = 0 \mid V = v)}$$

and

$$OR = \frac{P(Y^{a=1}=1)}{P(Y^{a=1}=0)} \Big/ \frac{P(Y^{a=0}=1)}{P(Y^{a=0}=0)} \ .$$

Present your answer in the form of a table with entries $A \times Y \times V$. Deduce that in general we cannot write OR as a weighted sum of OR_v with non-negative weights.

This property is referred to as the non-collapsibility of the odds ratio, and can be seen as a consequence of Jensen's inequality (an average over a non-linear function does not equal the function evaluated on the average). Thus, reporting odds ratios as effect measures arguably has undesirable features.

Exercise 4 (Positivity for standardization and IPW. Based on Technical Point 3.1 [1]). Consider a binary treatment A, a discrete baseline covariates L and an outcome Y. In the derivation of the weighted identification formula in the lectures, we showed that causal effect could expressed as the contrast

$$E[Y^{a=1} - Y^{a=0}] = E\left[\frac{I(A=1)}{\pi[A \mid L]}Y\right] - E\left[\frac{I(A=0)}{\pi[A \mid L]}Y\right] \tag{2}$$

under the assumption of conditional exchangeability, consistency. and positivity. The positivity condition is

$$P(A = a \mid L = l) > 0 \text{ for all } a \in \{0,1\} \text{ whenever } P(L = l) > 0 \ .$$

Next, we will consider what happens when positivity is violated. Suppose that there exists some a^*, l such that $P(A = a^* \mid L = l) = 0$ and $P(L = l) > 0$. Next, define $Q(a) = \{l : P(A = a \mid L = l) > 0\}$ to be the levels of L with positivity for treatment level a.

(a) Show that

$$E\left[\frac{I(A=a)Y}{\pi[A \mid L]}\right] = P(L \in Q(a)) \sum_{l \in Q(a)} E[Y \mid A = a, L = l] P(L = l \mid L \in Q(a)) \ .$$

(b) Explain why the naive contrast $E\left[\frac{I(A=1)}{\pi[A|L]}Y\right] - E\left[\frac{I(A=0)}{\pi[A|L]}Y\right]$ no longer has a causal interpretation under violation of positivity.

References

[1] Miguel Hernan and James M. Robins. *Causal Inference*. Chapman & Hall, 2018.

[2] Sander Greenland and Judea Pearl. Adjustments and their Consequences-Collapsibility Analysis using Graphical Models: Adjustments and their Consequences. *International Statistical Review*, 79(3):401–426, December 2011.

[3] Judea Pearl, James M. Robins, and Sander Greenland. Confounding and Collapsibility in Causal Inference. *Statistical Science*, 14(1):29–46, February 1999.