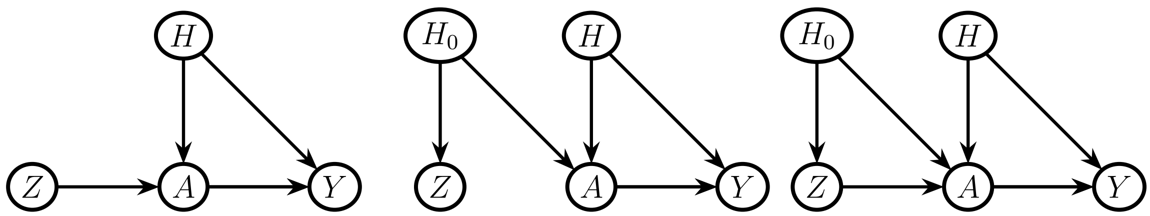
EXERCISES FOR RANDOMIZATION AND CAUSATION (MATH-336)

Exercise Sheet 11

Exercise 1 (Instrumental variables)**.** (From [1]) Consider an instrumental variable setting which is described by one of the following three DAGs.



(a) Can we use the main IV assumptions (1)-(3) to infer any (conditional) independencies between the observed variables A, Z, Y, that is, any factorization of the law $p(y, a, z)$ that describes the observed data? We reproduce the main IV assumptions below for convenience:
(1) $\text{cor}(Z, A) \neq 0$ (instrument strength)
(2) $Y^{z,a} = Y^a$ for all a, z (exclusion restriction)
(3) $Z \perp\!\!\!\perp Y^a$ for all a (unconfoundedness of Z).

(b) Consider the following structural equation model for Y:

$$Y = f_Y(A, H, \epsilon_Y) = h(\epsilon_Y)A + g(H, \epsilon_Y)\ . \tag{1}$$

The model does allow certain effect heterogeneity, because the individual level causal effect

$$Y^a - Y^{a'} = h(\epsilon_Y)(a - a')$$

is a random variable. The average causal effect is defined as

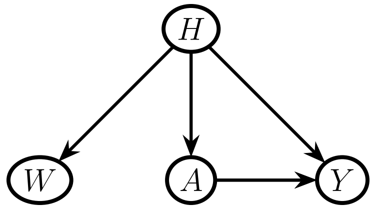
$$E[Y^a] - E[Y^{a'}] = E[h(\epsilon_Y)](a - a')\ .$$

Assume that the linear structural equation model Eq. 1 holds, that $Y^{a=0} \perp\!\!\!\perp Z$ and that $E[h(\epsilon_Y) \mid Z, A] = E[h(\epsilon_Y)]$. Show that the additive average causal effect is then given by

$$E[h(\epsilon_Y)] = \frac{\text{Cov}(Z, Y)}{\text{Cov}(Z, A)}\ .$$

(c) Assume that the model in Eq. 1 holds, and that $E[h(\epsilon_Y) \mid Z, A] = E[h(\epsilon_Y)]$. Show that then, there exists a constant β such that

$$E[Y \mid Z, A] - E[Y^0 \mid Z, A] = \beta A\ .$$



Exercise 2 (A sensitivity analysis)**.** Consider the treatment A, outcome Y, unmeasured variable H and measured pre-treatment variable W satisfying the graph below.

As we can see from the graph, both W and A are confounded for Y by H. Suppose that

$$E[Y^{a=0} \mid A=1] - E[Y^{a=0} \mid A=0] = E[W \mid A=1] - E[W \mid A=0] \ .$$

(a) Use this assumption to find an identification formula for $E[Y^{a=1} - Y^{a=0} \mid A=1]$ in terms of the observed data A, W, Y.
(b) Can we interpret this as an average total effect in the entire population?

Exercise 3 (Sensitivity analysis with IVs)**.** Consider a binary instrument Z, a binary treatment A and a binary outcome Y satisfying:
(1) Exclusion restriction: $Y^{z,a} = Y^a$
(2) IV exchangeability: $Y^a \perp\!\!\!\perp Z$
Show that under assumptions (1)-(2),

$$P(Y=0, A=1 \mid Z=0) + P(Y=1, A=1 \mid Z=1) \leq 1 \ .$$

Hint: Use the fact that $p(x_1, x_2 \mid x_3) \leq p(x_1 \mid x_3)$. Likewise, it can also be shown that

- $P(Y=0, A=1 \mid Z=0) + P(Y=1, A=1 \mid Z=1) \leq 1$
- $P(Y=0, A=1 \mid Z=1) + P(Y=1, A=1 \mid Z=0) \leq 1$
- $P(Y=0, A=0 \mid Z=1) + P(Y=1, A=0 \mid Z=0) \leq 1$

These inequalities can be used to falsify IV exchangeability assumption. With some more arguments, it is also possible to use the IV inequalities to obtain bounds on causal effects.

References

[1] Andrea Rotnitzky. BST 257 (Harvard T.H. Chan School of Public Health).