Exercice 1

1. In what dimensions is a simple random walk recurrent.
2. Give an example of a non irreducible Markov chain with a unique stationary distribution.
3. An irreducible Markov chain on $\{1, 2, 3, 4\}$ has statiionary distribution $(\frac{1}{4}, \frac{1}{8}, \frac{1}{8}\frac{1}{2})$. If $X_0 = 1$, what is the expectation of the number of visits to 2 before the chain returns to 1 for the first time?
4. Give an approximation to the matrix $\mathbf{P}^{500}$.
5. If π is a stationary distribution and x is a null recurrent site, then $\pi(x) = 0$. True or False. Justify your answer.
6. Define a stopping time for a Markov chain and give a random time that is not a stopping time.

1) THE SIMPLE RANDOM WALK IS RECURRENT IN DIMENSIONS 1 e 2. (VIA OUR ~~CALCULATION ESTIMATION~~ BOUNDS ON p_{00}^{2n} VIA STIRLINGS FORMULA

2) $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ ON $I = \{0, 1\}$ ONLY EQUILIBRIUM IS ~~(0, 1)~~. $\lambda_0 = 0$ $\lambda_1 = 1$

IT IS NOT IRREDUCIBLE AS $1 \not\to 0$.

3) THE CHAIN IS IRREDUCIBLE SO THIS IS $\frac{\pi(2)}{\pi(1)} = \frac{1/8}{1/4} = 1/2$.

4) FOR ME IT IS $\begin{pmatrix} 1/4 & 1/8 & 1/8 & 1/2 \\ 1/4 & 1/8 & 1/8 & 1/2 \\ 1/4 & 1/8 & 1/8 & 1/2 \\ 1/4 & 1/8 & 1/8 & 1/2 \end{pmatrix}$ IF THE CHAIN IS APERIODIC.

IF NOT THEN IT COULD BE

1/2 1/4 1/4 0
1/2 1/4 1/4 0
1/2 1/4 1/4 0.
0 0 0 1

5) TRUE. IF x IS NULL RECURRENT THEN THE LONG RUN PROPORTION OF TIME x IS OCCUPIED IS 0 FOR ANY INITIAL ~~CONDITIO~~ DISTRIBUTION INCLUDING THE STATIONARY DISTRIBUTION π

6) FOR A MC $(X_n)_{n \geq 0}$ T IS A STOPPING TIME IF IT IS A R.V. TAKING VALUES IN $\{0, 1, 2, \ldots\} \cup \{\infty\}$ AND SO THAT $\{T = n\}$ ~~E~~ AMOUNTS TO $\{(X_0 \ldots X_n) \in A_n \subseteq I^{n+1}\}$ $\forall n$.

AN INTEGER VALUED ~~STOPPING TIME~~ NON STOPPING TIME IS FOR CHAIN $\begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$ ON $I = \{0, 1\}$ $X_0 = 0$ T = LAST TIME $X_n = 0$

Exercice 2

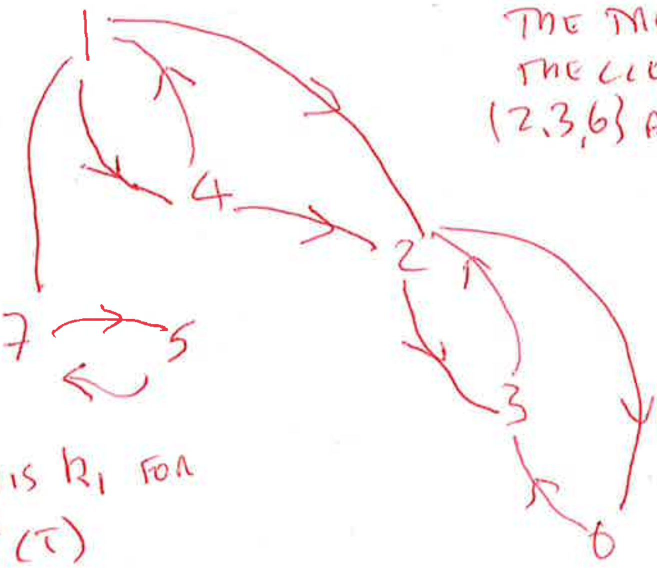
Consider the transition matrix $\mathbf{P}$ =

$$\begin{pmatrix} 1/3 & 2/9 & 0 & 1/3 & 0 & 0 & 1/9 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/2 & 1/4 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1/4 & 0 & 0 & 3/4 & 0 \\ 0 & 0 & 0 & 0 & 1/5 & 0 & 4/5 \end{pmatrix}$$

with state space $S = \{1, 2 \cdots 7\}$ and let $(X_n)_{n\geqslant 0}$ be corresponding Markov chain.

1. List the transient states and the closed communicating classes subsets.
2. Give a good approximation to $\mathbf{P}^{100}_{17}$.
3. For $\tau = \inf\{n \geqslant 0 : X_n \notin \{1, 4\}\}$, find $E^1[\tau]$.

1)



THE TRANSIENT STATES ARE 1,4
THE CLOSED IRREDUCIBLE SETS ARE
{2,3,6} AND {5,7}

2) $P^{100}_{1,7} \approx h_1 \pi(7)$

FOR $h_i = P^i(T_7 < \infty)$ $i = 1,4$

AND π EQUILIBRIUM ON $\{5,7\}$:

(i) CALCULATE h_1

$h_1 = \frac{1}{3} h_1 + \frac{2}{9} h_2 + \frac{1}{3} h_4 + \frac{1}{9} h_7$

$= \frac{1}{3} h_1 + \frac{1}{3} h_4 + \frac{1}{9}$

$h_4 = \frac{1}{2} h_1 + \frac{1}{4} h_2 + \frac{1}{4} h_4$

$= \frac{1}{2} h_1 + \frac{1}{4} h_4$

$\Rightarrow 2h_1 = h_4 + \frac{1}{3}$

$\Rightarrow h_4 = 2h_1 - \frac{1}{3} = \frac{1}{2} h_1 + \frac{1}{4}(2h_1 - \frac{1}{3})$

$\Rightarrow h_1 = \frac{1}{4}$

(ii) CALCULATE π_7:

$\pi_5 = \frac{2}{3}\pi_5 + \frac{1}{5}\pi_7$

$\Rightarrow \frac{1}{3}\pi_5 = \frac{1}{5}\pi_7 \Rightarrow \pi_7 = 5/8$

ANSWER
$P^{100}_{1,7} \approx \frac{1}{4} \cdot \frac{5}{8}$

3) ANSWER IS h_1 FOR
$h_i = E^i(\tau)$

$h_1 = 1 + \frac{1}{3} h_1 + \frac{1}{3} h_4$

$h_4 = 1 + \frac{1}{2} h_1 + \frac{1}{4} h_4$

$2h_1 = 3 + h_4$

$= 3h_4 - 4 \Rightarrow h_4 = 7/2$

$\Rightarrow h_1 = \frac{3 + 7/2}{2} = \boxed{\frac{13}{4}}$

Exercice 3

A knight moves on a reduced chess board (4 by 4 subsquares instead of 8 by 8) : at each turn he chooses a possible move .

1. In equilibrium is the resulting markov chain reversible ?
2. If the knight starts in a corner, what approximately is the probability that after 1000 moves he is again in the same corner ?
3. In the time interval $[0, 1000]$, how many visits (approximately) will he have made to this corner ?
4. The $16 = 4 \times 4$ positions contain 12 positions on the border (which touch the exterior and which are thus a distance 1 from the exterior) and 4 central points of distance 2 from the exterior. After 1000 moves what is approximatively the average of distance from the extérior for the knight ?

1) YES IT IS A RANDOM WALK ON A GRAPH

2)

2	3	3	2
3	4	4	3
3	4	4	3
2	3	3	2

STATIONARY DISTN IS $\frac{d_x}{48}$

BY CONVERGENCE THM

PROB $\longrightarrow 2 \ \frac{2}{48}$ (PERIOD $= 2$) $= \frac{1}{12}$

3) APPROXIMATELY $10{,}000 \times \pi_x = 10{,}000 \ \frac{2}{48}$

4) THE AVERAGE DISTANCE SHOULD BE APPROXIMATELY (BY ERGODICITY) $\int$ THE INTEGRAL OF DISTANCE WRT EQUILIBRIUM DISTN

i.e. $1 \times \frac{32}{48} + 2 \times \frac{16}{48}$