0) NOTE CORRECTION $W_5 = \sum_{i \leq J_5 \leq t} I_{M_i = 5}$ (NOT 6 !)

a) BY THEOREM 2.4.6 THIS ~~IS~~, $P(X_1 = 1 \mid X_2 = 3)$ THE PROBABILITY THAT FOR 3 i.i.d. $U[0,2]$ r.v.s EXACTLY 1 IS IN $[0,1]$. THAT IS $P(\mathrm{Bin}(3, \tfrac{1}{2}) = 1) = \binom{3}{1}(\tfrac{1}{2})^2(\tfrac{1}{2}) = 3/8$

THE ANSWER FOR $P(Y_1 = 1 \mid X_1 = 3)$ IS EXACTLY THE SAME, 3/8, SINCE THE POISSON PROCESS RATE IS NOT RELEVANT BY 2.4.6.

b) $(Y_t)_{t \geq 0}$ AND $(X_t - Y_t)_{t \geq 0}$ ARE INDEPENDENT, POISSON RATE 1 $(= 2 \times \tfrac{1}{2})$

SO $P(X_t = 2Y_t) = P(X_t - Y_t = Y_t)$

$= \sum_{r=0}^{\infty} P(X_t - Y_t = r) P(Y_t = r)$ BY INDEPENDENCE

BOTH Y_t, $X_t - Y_t$ ARE POISSON r.v.s SO THIS IS

$$\sum_{r=0}^{\infty} \left(e^{-t} \frac{t^r}{r!}\right)^2 = e^{-2t} \sum_{r=0}^{\infty} \frac{t^{2r}}{r!\,r!}$$

FOR THE SECOND PART. $P(X_t = 2Y_t \mid X_t \text{ IS EVEN})$ IS SIMPLY

$$\frac{P(X_t = 2Y_t)}{P(X_t \text{ IS EVEN})}$$ AS $\{X_t = 2Y_t\} \subseteq \{X_t \text{ IS EVEN}\}$

SO IT REMAINS TO ~~SHOW~~ CALCULATE $P(X_t \text{ IS EVEN})$

$$= \sum_{r=0}^{\infty} P(X_t = 2r) = e^{-2t} \sum_{r=0}^{\infty} \frac{(2t)^{2r}}{(2r)!}$$

$$= e^{-2t} \left(\frac{e^{2t} + e^{-2t}}{2} \right)$$

$$= \frac{1 + e^{-4t}}{2}$$ (NOTE THIS ACCORDS WITH KNOWN BEHAVIOUR AT $t = 0$ AND $t \to \infty$)

So $P(X_t = 2Y_t \mid X_t \text{ is even}) = \dfrac{e^{-2t}\sum_{r=0}^{\infty}\frac{t^{2r}}{r!r!}}{\frac{1+e^{-4t}}{2}}$

$$= \frac{\sum_{r=0}^{\infty} t^{2r}/r!r!}{\cosh(2t)}$$

FINALLY BY SYMMETRY (AS $X_t - Y_t \overset{D}{=} Y_t$ AND IS INDEPENDANT OF IT

$$P(X_t > 2Y_t \mid X_t \text{ is even})$$
$$= \frac{1}{2} P(X_t \neq 2Y_t \mid X_t \text{ is even}) = \frac{1}{2}\left(1 - \frac{\sum_{r=0}^{\infty} t^{2r}/r!r!}{\cosh(2t)}\right)$$

c) THIS FIRST PART RESTS ON THE FACT THAT JUMP TIMES FOR Z ARE DERIVED FROM JUMP TIMES FOR Y WITH PROBABILITY $1/3$.

THUS $P(Z_1 = 1 \mid Y_2 = 5) = \sum_{k=0}^{5} P(Z_1 = 1 \mid Y_2 = 5, Y_1 = k) P(Y_1 = k \mid Y_2 = 5)$

$$= \sum_{k=0}^{5} P(Z_1 = 1 \mid Y_2 = 5, Y_1 = k)\binom{5}{k}(1/2)^5$$

$$= \sum_{k=1}^{5} P(Z_1 = 1 \mid Y_1 = k)\binom{5}{k}(1/2)^5$$ AS ONLY Y_1 IS RELEVANT TO Z_1 AND IF $Y_1 = 0$ Z_1 CANNOT BE 1.

$$= \sum_{k=1}^{5}\binom{5}{1}(1/3)(2/3)^{k-1}\binom{5}{k}(1/2)^5$$

$P(W_1 = 1 \mid Y_2 = 5)$ IS $P(W_1 = 1)$ AS (WITH CORRECT DEFINITION OF W) W, Y ARE INDEPENDANT.

$P(W_1 = 1 \mid Y_2 = 5)$ IS THEREFORE $e^{-1/3}(1/3)$.

1). THE SEMI GROUP $(P(t))_{t\geq 0}$ SATISFIES

BACKWARD EQUATION: $P(0)=Id \quad P'(t) = QP(t)$

SO $P_{12}'(t) = -2P_{12}(t) + P_{22}(t) + P_{32}(t) \qquad P_{12}(0)=0.$

FORWARD EQUATION $\quad P(0)=Id \quad P'(t) = P(t)Q.$

SO $P_{12}'(t) = -P_{12}(t) + P_{11}(t) + P_{13}(t) \qquad P_{12}(0)=0$

$$P_{12}(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}(Q^r)_{12}.$$

2) $$Q = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

SO $0 \xrightarrow{2} 1 \quad 0 \xrightarrow{1} 2 \quad 1 \xrightarrow{1} 2$

BY FORWARD EQUATION

$P_{01}'(t) = P_{00}(t) - P_{01}(t)$ BUT $P_{00}(t) = e^{-2t}$

SO $P_{01}'(t) = e^{-2t} - P_{01}(t)$

$P_{01}'(t) + P_{01}(t) = e^{-2t}.$

$e^t P_{01}'(t) + e^t P_{01}(t) = e^{-t}.$

SO $(e^t P_{01}(t)) = C - e^{-t}.$ AS $P_{01}(0)=0$

$= 1 - e^{-t}.$

SO $P_{01}(t) = e^{-t} - e^{-2t}.$

SO $P_{02}(t) = 1 - P_{00}(t) - P_{01}(t)$

$= 1 - e^{-2t} - (e^{-t} - e^{-2t}) = 1 - e^{-t}$

AND IMMEDIATELY

$P_{10}(t) = 0 \quad P_{11}(t) = e^{-t}$ SO $P_{12}(t) = 1 - e^{-t}.$

$P_{21}(t) = P_{20}(t) = 0 \quad P_{22}(t) = 1$

3) $\mathbb{P}^i(X$ MAKES AT LEAST TWO JUMPS IN $[0,t])$

$$= \sum_{j\neq i} \mathbb{P}^i(X \text{ MAKES AT LEAST TWO JUMPS IN } [0,t] \cap Y_1 (= X_{J_1} = j))$$

$$\leq \sum_{j \neq i} \mathbb{P}^i(X \text{ MAKES A FIRST JUMP TO } j \text{ IN TIME } [0,t])\, \mathbb{P}^j(X \text{ MAKES A JUMP IN } [0,t])$$

So $\dfrac{\mathbb{P}^i(X \text{ MAKES AT LEAST TWO JUMPS IN } [0,t])}{t}$

$$\leq \sum_{j\neq i} \frac{\mathbb{P}^i(X \text{ MAKES A FIRST JUMP TO } j \text{ IN TIME } [0,t])}{t}\, \mathbb{P}^j(X \text{ MAKES A JUMP IN } [0,t]).$$

NOW $\dfrac{\mathbb{P}^i(X \text{ MAKES A FIRST JUMP IN TIME } [0,t] \text{ TO } j)}{t}$

$$= \frac{(1-e^{q_i t})}{t}\pi_{ij} \leq \frac{q_i t}{t}\pi_{ij} = q_{ij}.$$

SO $0 \leq \displaystyle\sum_{j\neq i} \frac{\mathbb{P}^i(X \text{ MAKES A FIRST JUMP TO } j \text{ IN } [0,t])}{t}\, \mathbb{P}^j(X \text{ MAKES A JUMP IN } [0,t].)$

$$\leq \sum_{j\neq i} q_{ij}\, \mathbb{P}^j(X \text{ MAKES A JUMP IN } [0,t])$$

BUT $\sum q_{ij} = q_i < \infty$ AND $\forall j$ $\mathbb{P}^j(X \text{ MAKES A JUMP IN } [0,t]) \longrightarrow 0$ AS $t \to 0$.

So BY BOUNDED (OR DOMINATED (OR MONOTONE)) CONVERGENCE

$$\frac{\mathbb{P}^i(X \text{ MAKES TWO JUMPS IN } [0,t])}{t} \xrightarrow{t\to 0} 0$$

THAT IS $\mathbb{P}^i(X \text{ MAKES TWO JUMPS IN } [0,t])$ IS $o(t)$

4) (i) IF $q_{i\,i+1} = i+1$ AND $q_{i0} = 1$ $q_{ij} = 0$ $\forall j \notin \{0, i+1\}$

MEANS ONLY POSSIBLE JUMP FROM i IS ($\forall i$ $i>0$)

TO $i+1$ WP $\frac{i+1}{i+2}$.

TO 0 WP $\frac{1}{i+2}$.

AS $\sum_{i=0}^{\infty} \frac{1}{i+2} = \infty$ WE HAVE BY BOREL CANTELLI II (OR DIRECT CALCULATION) THAT $\mathbb{P}^0(X$ RETURNS TO 0 AGAIN$) = 1$

THAT IS 0 IS RECURRENT. FOR THE JUMP CHAIN. SO

~~BY THEOREM 2.7.1 (iii) STARTING FROM~~.

SO BY IRREDUCIBILITY ALL SITES ARE RECURRENT FOR JUMP CHAIN

SO BY THEOREM 2.7.1 (iii) THE PROBABILITY OF AN EXPLOSION $= 0$ (IRRESPECTIVE OF INITIAL POSITION)

(ii) THE PROCESS IS A PURE BIRTH PROCESS (SO THE JUMP CHAIN IS TRIVIAL) WE TAKE $\mathcal{E}xp((i+1)^3)$ TO PASS FROM STATE i TO STATE $i+1$.

AS $\sum \frac{1}{(i+1)^3} < \infty$ (AND THEOREM 2.3.2)

THE SUM OF ALL HOLDING TIMES (IRRESPECTIVE OF INITIAL CONDITION) IS FINITE. THAT IS WP 1 THERE IS AN EXPLOSION.

(iii) THIS CASE IS LIKE CASE (i) (BUT SPEEDED UP)

THE JUMP RATE AT i HAS CHANGED FROM $i+2$ TO $(i+1)^3 + (i+1)^2 = (i+1)^2(i+2)$

BUT THE JUMP CHAIN IS STILL $\Pi_{01} = 1$.

$\Pi_{i\,i+1} = \frac{i+1}{i+2}$ $\Pi_{i0} = \frac{1}{i+2}$.

SO THEOREM 2.7.1 (iii) STILL APPLIES

SO a.s. ($\forall$ INITIAL VALUE X_0) THERE IS NO EXPLOSION

5) CONSIDER THE JUMP CHAIN $(Y_n)_{n\geq 0}$: $Y_n = X_{J_n}$.

WE HAVE $(|Y_n|)_{n\geq 0}$ IS JUST THE MARKOV CHAIN ON $\mathbb{N}$ WITH $P_{01}=1$ $P_{i\,i-1}=1/3$ $P_{i\,i+1}=2/3$ $\forall i>0$.

AS SUCH (SEE EX 1.3.3 PAGE 15) $\forall$ $i\in\mathbb{Z}\setminus\{0\}$ THE PROB $\mathbb{P}^i(H_0<\infty)=(1/2)^{|i|}\leq 1/2$ FOR $i\neq 0$ AND $\forall i$

$$\mathbb{P}^i(T_i<\infty)=1/2 \quad \forall\, \cancel{i\neq 0}\ i \qquad (*)$$

SO IT FOLLOWS ~~FAUTOME~~ THAT 0 IS TRANSIENT AND STARTING FROM 0 THE JUMP CHAIN MUST GO TO $\pm\infty$ W.P. 1

BY SYMMETRY

$$\mathbb{P}^0(\lim Y_n=\infty)=1/2=\mathbb{P}^0(\lim Y_n=-\infty)$$

BY OUR CONSTRUCTION, IF $\lim Y_n=-\infty$ THEN THE HOLDING TIMES OF $Y_0, Y_1 \ldots Y_n \ldots$ WILL CONTAIN INFINITELY MANY WITH $q_{Y_j}\leq 1$ SO NECESSARILY BY CONDITIONAL INDEPENDENCE THERE CAN BE NO EXPLOSION IN THIS CASE.

ON THE OTHER HAND FOR $i>0$ $q_i=e^i$ WHICH IS GOING TO ∞ VERY QUICKLY. I HAVE $\forall i$ BY $(*)$ AND THE STRONG ~~STRONG~~ MARKOV PROPERTY $\mathbb{E}^0(\#\text{ OF VISITS TO } i)\leq 2$.

$$\text{SO } \mathbb{E}^0(\text{TIME SPENT BY } X_t \text{ IN }]0,\infty[)\leq\sum_{i=1}^{\infty}2\cdot\frac{1}{e^i}<\infty$$

SO A.S. TIME SPENT BY CHAIN X_t IN $]0,\infty[$ $<\infty$

SO IF $\lim_{n\to\infty} Y_n=\infty$ THEN TIME SPENT IN $]-\infty,0]$ IS ALSO FINITE AND SO TIME SPENT IN $\mathbb{Z}$ IS FINITE IE THERE IS AN EXPLOSION

5 CONTD. SO THE EVENT {THERE IS AN EXPLOSION} IS

IS THE SAME AS THE EVENT $\{\lim_{n\to\infty} Y_n = \infty\}$

SO $\mathbb{P}^0(\text{EXPLOSION}) = 1/2$.

IF $Y_0 = 1$ WE HAVE

$$\mathbb{P}^1(\text{EXPLOSION}) = \mathbb{P}^1(\text{EXPLOSION} \cap \{T_0 < \infty\}) + \mathbb{P}^1(\text{EXPLOSION} \cap \{T_0 = \infty\}).$$

$$= \mathbb{P}^1(\text{EXPLOSION} \cap (T_0 < \infty)) + \mathbb{P}^1(T_0 = \infty)$$

$$= \mathbb{P}^1(\text{EXPLOSION} \cap \{T_0 < \infty\}) + 1/2.$$

BUT THE CHAIN "RESTARTS AT T_0" SO

$$\mathbb{P}^1(\text{EXPLOSION} \cap \{T_0 < \infty\})$$

$$= \mathbb{P}^1(\cancel{\text{ex}}\, T_0 < \infty)\, \mathbb{P}^1(\text{EXPLOSION} \mid T_0 < \infty)$$

$$= 1/2 \qquad \mathbb{P}^0(\text{EXPLOSION})$$

$$= 1/2 \qquad 1/2 \qquad = 1/4$$

SO $\mathbb{P}^1(\text{EXPLOSION}) = 3/4$